Time	Activity			Venue	Staff
10:30	Arrival – Registration and Refreshments			Roscoe Foyer	
11:00	Opening Plenary - 10 mins UoM - 30 mins Dr. Rose Wagstaffe			Roscoe Theatre B	UoM
11:40	How to use Integral and EdX				MEI
11:55	Move to workshop rooms				
12:05	Icebreakers in Mentoring groups (+ choose session times)				Mentors (UoM)
	Group	Tutor Groups	Room		
	A	1 2 3	Roscoe 1.001 Roscoe 1.003 Roscoe 1.007		
	B	4 5	Roscoe 1.009 Roscoe 1.010		
12:50	Lunch (quiet/prayer room Roscoe 2.4)		Group A	Roscoe Foyer	
			Group B	Roscoe Foyer	
1:30	1st Workshop				MEI
	Workshop Name	Groups	Room		
	Complex Numbers	A	Roscoe 2.2		
	Matrix Multiplication	B	Roscoe 2.3		
2:30	Swap workshop rooms				
2:35	2nd Workshop				MEI
	Workshop Name	Groups	Room		
	Complex Numbers	B	Roscoe 2.2		
	Matrix Multiplication	A	Roscoe 2.3		
3:35	Move to main room				
3:40	Closing plenary and questions			Roscoe Theatre B	UoM
4:15	Event End				

Emergency Information

In the event of an emergency please contact the event mobile on your lanyard or call:

University Security – 0161 306 9966

Event Mobile – 07467 339519

If you are lost or separated from your group, go to Roscoe Foyer.

Further mA*ths Online Programme – Masterclass 1

This is a time for you to get to know your mentor and the other people in your group.

There are some maths puzzles and problems to try if you want.

You might like to start by introducing yourself: Tell your group who you are, where you are from and why you wanted to take part in this course.

You will find that your experience of Further Maths may well be different from others in your group. Different schools take different approaches to teaching Further Maths courses so you may find that you know different things to the rest of your group.

We'd like you to answer a quick questionnaire to let your mentor know what you have covered so far. All of the skills/techniques and ideas in the questionnaire are required for the workshops. It will help your mentor if they know how much additional information and help they will need to give.

You can tick off on this sheet if you wish but the better thing to do is discuss with your mentor and group what you have and haven't covered.

Don't be worried about admitting that you haven't covered something, the MOOC part of the course will provide you with a lot of the content required for Further Maths.

Questionnaire

Part 1 Complex Numbers

Which of the following skills/techniques/ideas from complex numbers have you either covered or seen. Say if you have covered the content, heard of it or have not yet seen it.

Skill/technique/idea	Covered it	Heard of it	Not yet seen
The idea of an imaginary number (from $i = \sqrt{-1}$)			
The idea of a complex number ($z = x + yi$)			
Adding and subtracting complex numbers			
Multiplying and dividing complex numbers			
The conjugate of a complex number			
Representing complex numbers on an Argand diagram			
The modulus and argument of a complex number			
Loci on the Argand diagram			

Part 2 Matrices

Skill/technique/idea	Covered it	Heard of it	Not yet seen
The order (dimension) of a matrix			
Adding and subtracting conformable matrices			
Multiplying a matrix by a constant			
Multiplying conformable matrices (matrix multiplication)			
Linear transformations using 2×2 matrices			
The determinant of a 2×2 matrix			

If you are running out of things to talk about, have a look at the problems on the next page. These are a set of mathematical problems that were selected as favourites by various mathematicians from MEI. Pick one or two and have a go.

If you do try any of these problems you should talk about them with others in your group. Don't worry if you don't manage to find an answer to any of the problems, it is not an examination, just an icebreaker to get you thinking before you go into the rest of the day.

Favourite Problems

Here is a set of 6 problems that some mathematicians chose as their favourite problems that could be solved using only mathematics up to A level.

## My favourite problem is...
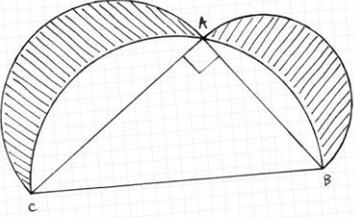
This problem looks complicated but requires only a few simple techniques to reveal a remarkable solution and a 2500 year old theorem.

Triangle ABC has a right-angle at A.

Semi-circles are drawn with BA, AC and BC as diameters as shown.

Given that AC = 8 and BC = 10, write down the value of one third of the total shaded area.

From Senior Team Mathematics Challenge (UKMT/FMSP)

My favourite problem is...
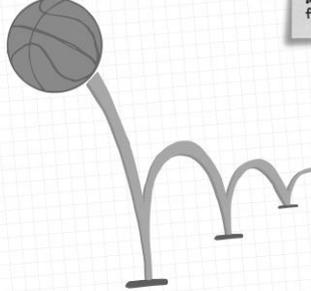
The first part can be solved with decimals and percentages. It leads on to exploring a sequence of numbers with powers and using logarithms for solving equations.

A ball is dropped and bounces up to a height that is 75% of the height from which it was dropped. It continues to bounce, each time to a height that is 75% of the previous height.

How many bounces does it make before it bounces up to less than 10% of the original height? How many for less than 1%?

My favourite problem is...
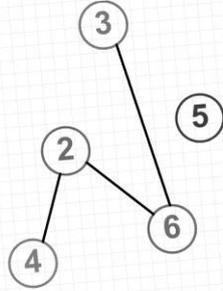
This problem is easy for anyone to start but gets increasingly complicated and requires a proof to know you have the best answer.

Starting with the number 2, integers are added, in order, so that any two numbers are joined if one of them is a factor of the other. The diagram shows the connections for the numbers 2 to 6.

What is the maximum number that can be reached if no lines joining two numbers are allowed to cross?

Based on a problem from NRICH

My favourite problem is...
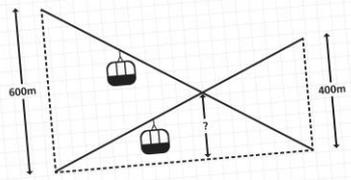
It's amazing that the height of the crossing can be calculated at all in this problem, surely it is necessary to know how far apart the two ground level stations are?

Two straight wires for cable cars are shown in the diagram. One car reaches a vertical height of 600m and the other a vertical height of 400m as shown.

At what height do the wires cross?

From Professor Stewart's Cabinet of Mathematical Curiosities by Ian Stewart

MY FAVOURITE PROBLEM IS...

This problem can be solved in several ways, using techniques from GCSE or A level and even beyond that! The surprising result fascinated Nobel Prize winning Physicist Richard Feynman.

1. Start with any triangle.

2. Mark a point one third of the way along each edge moving anticlockwise from each corner.

3. Join these points to the 'opposite' corner to make a new triangle.

4. What fraction of the original triangle is the new triangle?

Based on a problem in The Mathematical Gazette

My favourite problem is...

This deceptively simple problem can lead to complicated algebra but it does have a concise and very satisfying solution

A fisherman fishing from the edge of a pond hooks a fish at position A and begins to reel it in. He reels in 1m of line bringing the fish closer to the shore, parallel to the surface of the water.

Is the fish now closer to the shore by:
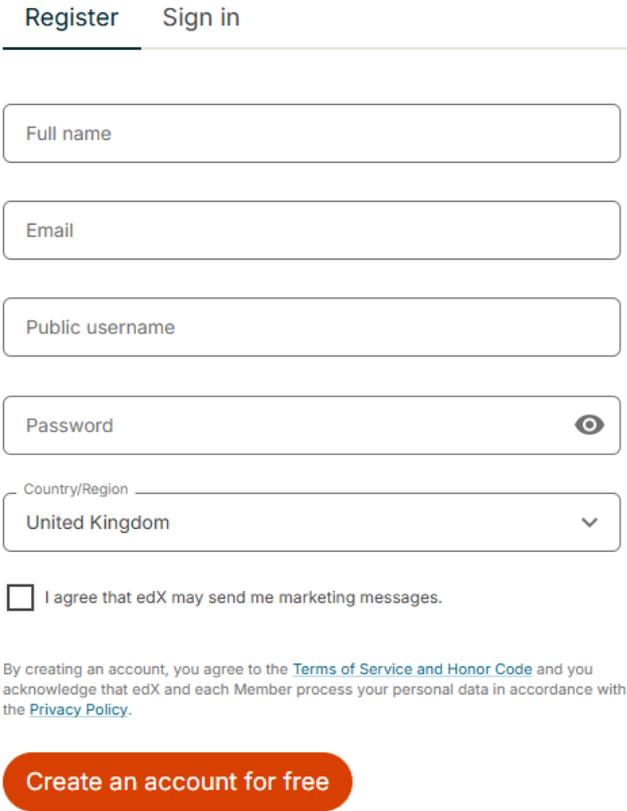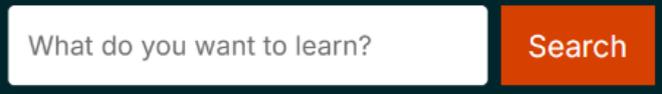More than 1m; Less than 1m; or Exactly 1m?

Pick one or two and give them a go. There are no solutions given – can you convince yourself that whatever answer you come up with is the correct one.

Do you have a favourite maths problem – maybe it has a really clever trick or a really "cute" answer.

Can you describe your favourite problem to the rest of your group?
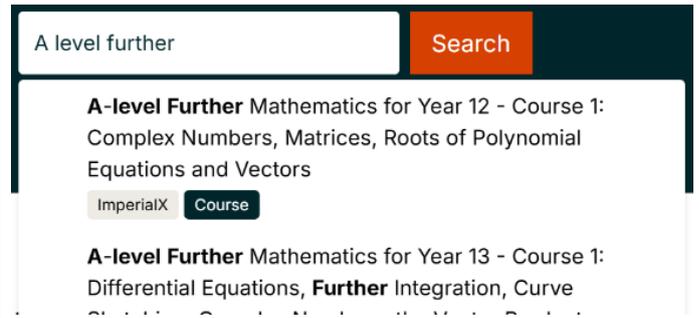
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Further mA*ths Online Programme – Masterclass 1

Registering with EdX

Instruction	Screenshot
Go to https://www.edx.org/ From the top right-hand corner of the webpage, click on 'Register for free'.	**Register for free**
You will be asked to fill in a short form. Although you can sign in with a Facebook, Google+ or Microsoft account, we ask that you create a new account using the form. This will allow us to track your progress through the course. Fill in all of the spaces. Be very careful to use a working email and enter it carefully. Your *Public Username* cannot contain spaces. Use something sensible. It must be unique. The *password* needs both numbers (at least 1) and letters. Untick the marketing box if you wish, and click to *'Create an account for free'*.	**Register** **Sign in** Full name Email Public username Password Country/Region United Kingdom ☐ I agree that edX may send me marketing messages. By creating an account, you agree to the Terms of Service and Honor Code and you acknowledge that edX and each Member process your personal data in accordance with the Privacy Policy. **Create an account for free**
You will be sent an email message to verify your account. If you can't access your emails at the moment, don't worry, you can verify it when you get home – don't forget.	**Confirm my email**
EdX will give you the following message: Which should not come as a surprise!	What do you want to learn? **Search**

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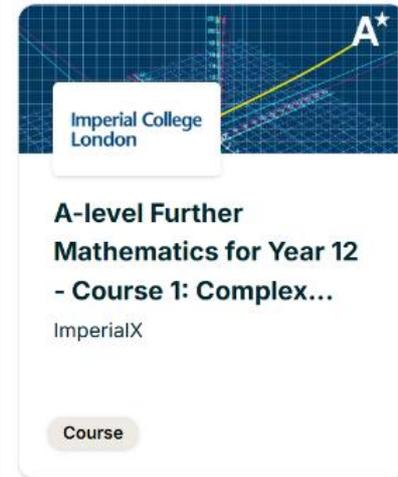|	
The use the search feature to search for '*A Level Further Mathematics*'. You will actually probably see the ImperialX courses appearing Click '*Search*' to see all results.	**A level further**	**Search** **A-level Further** Mathematics for Year 12 - Course 1: Complex Numbers, Matrices, Roots of Polynomial Equations and Vectors ImperialX Course **A-level Further** Mathematics for Year 13 - Course 1: Differential Equations, **Further** Integration, Curve
You are looking for the course: 'A-level Further Mathematics for Year 12 – Course 1...' Click on the course.	Imperial College London **A-level Further Mathematics for Year 12 - Course 1: Complex...** ImperialX Course
Click the 'Audit course' button under box on the right hand side. On the next page, scroll past the box asking for money and click 'Audit course' again **Important:** When you 'Audit' a course on edX you will still get the full experience. You will get messages asking you to pay. **Do not pay on edX.** You will get a certificate directly from Imperial College and not through edX.	**Starts Aug 05** Ends Sep 01 **Advance your career** **8,317** learners enrolled ⇄̌ Audit course

Well done, you have registered on the course in edX.

Make sure you remember your password.

An introduction to the online platforms

Further mA*ths Year 12 Day 1

1

Why are we using 2 online platforms?

The online course is entirely hosted in EdX

All of the exercise and assessments are automatically marked and recorded by EdX

But…EdX does not include a classroom for mentoring sessions.

2

What each platform is used for

Hosts the online course

All the videos, reading sections, exercises and assessments are on EdX.

The day to day platform for the course.

The support platform

Your mentor's classroom. Your group forum.

Where you go to get help when you are stuck!

3

The Masterclasses

The workshops in this masterclass have more activities than you will be able to complete.

You can continue with the activities on Integral. All of today's workshops are set up for you to continue working through.

Today each workshop will start with an introduction. You will then go into your mentor groups to work through the activities.

4

Using Integral

The first time you log in, you will be asked to change your password.
Make sure you remember your new password when you do this.
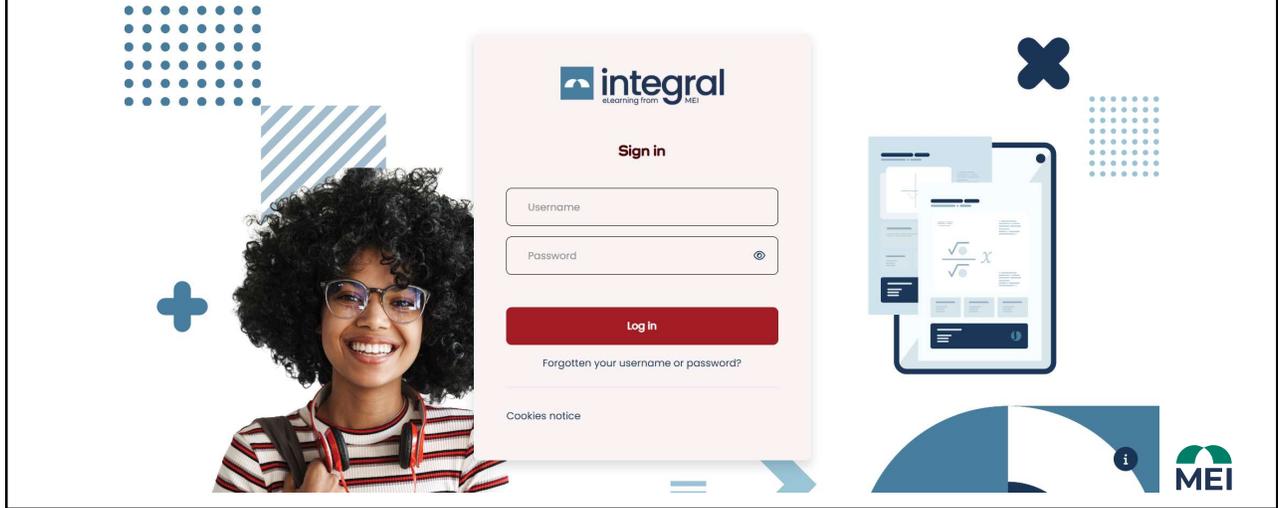
5

Using Integral

You should reach the dashboard. If you don't, click here

Access the course using either of these

6

3

Course page

Announcements

Discussion forum
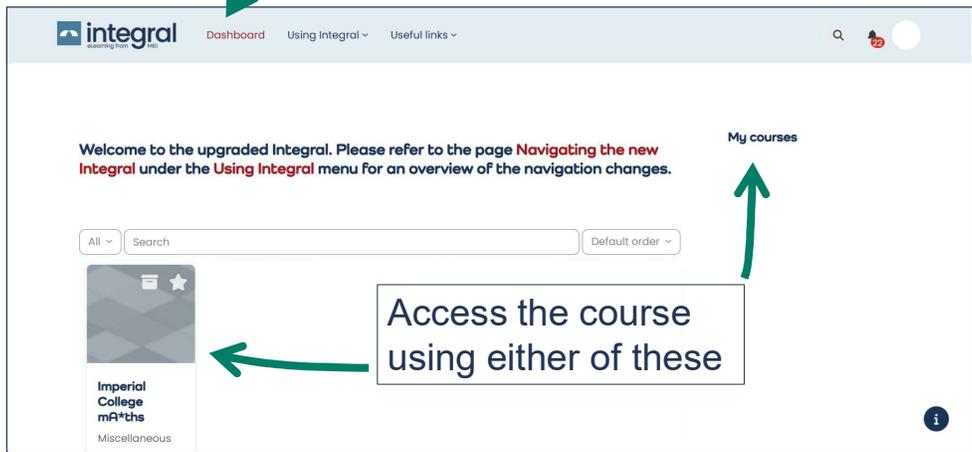
Online classroom

You can ask questions using the forum. A forum post will only be seen by your mentor and your group

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7

Using Integral - forum

Forum posts are read and moderated by the course staff.

The forum looks like this

Imperial mA*ths / Discussion forum

Discussion forum

Forum   Settings   Advanced grading   Subscriptions   Reports

Search forums   **Add discussion topic**   Unsubscribe from forum

Click here to post a new message

Separate groups   All participants

Read posted messages here

1 2 3 4 5 »

Discussion	Group	Started by	Last post ↓	Replies	Subscribe
☆ Closing message	2322-Y12Group1...	12 May 2025	12 May 2025	0	

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8

4

Using Integral – posting a message

To post a message

You have to put a subject

Type your
message here

You can enter maths
using an equation editor

Click here to post it!

9

Mentor classrooms

You access your mentor's classroom here

Announcements

Discussion forum

Online classroom

A headset or even just normal headphones will make the
classroom experience much better

10

Workshop materials

Announcements

Discussion forum

Online classroom

Mentor Area

A Level Further Mathematics for Year 12

Today's workshops are here

A Level Further Mathematics for Year 13

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11

Using Integral – Workshop pages

The workshops

Workshop 1 nth roots and complex numbers

H5P Revision – Imaginary and complex numbers

H5P Activity 1 – Raising complex numbers to a power

PDF Activity 1 pdf

H5P Activity 2 – Roots of 1 on an Argand diagram

PDF Activity 2 pdf

H5P Activity 3 – the relationship between the roots of unity

PDF Activity 3 pdf

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12

Registering with EdX

There are complete "from scratch" instructions in Integral.

If you have studied a course in EdX before, you will just need to add the new course to your portfolio.

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13

Registering with EdX

Go to https://www.edx.org/

Click on **Register for free**

You will be asked to fill in a short form

You can sign in with an Apple, Facebook, Google or Microsoft account, but we ask that you create a new account using the form.

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Registering with EdX

Your Public Username cannot contain spaces. Use something sensible. It has to be unique.

The password needs both numbers (at least one) and letters.

Untick the marketing box if you wish.

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15

Registering with EdX

The use the search feature to search for A Level Mathematics

Click SEARCH

A level further | Search

You are looking for the course A Level Further Mathematics for Year 12 - Course 1…

Course

Imperial College London

A*

A-level Further Mathematics for Year 12 - Course 1: Complex…

ImperialX

7 weeks to complete
Intermediate

MEI

16

Registering with EdX

Click the 'Audit course' button under box on the right-hand side.

On the next page, scroll past the box asking for money and click 'Audit course' again.

Important:

When you 'Audit' a course on edX you will still get the full experience.

Do not pay on edX. You will get a certificate directly from Imperial College and not through edX.

Starts Aug 05
Ends Sep 01

Advance your career

8,317 learners enrolled

Audit course

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Time to get started...

You are going to move in your mentor groups.

There are icebreaker starters to work on with your mentor and the others in the group.

Your mentor will also be checking your availability for their support sessions.

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18

Add details of icebreaker rooms here

MEI

19

n$^{\text{th}}$ roots and complex numbers

Imaginary and complex numbers

For the activities in this workshop you will need to know

- That imaginary numbers are based on the square root of -1

- That the notation i is used for $\sqrt{-1}$ so $i = \sqrt{-1}$

- That any imaginary number can be written in terms of i e.g. $\sqrt{-4} = 2i$

- That complex numbers are numbers with a **real** and **imaginary** part (either of which could be 0)

- That complex numbers can be written in the form $z = x + yi$ where x is the real part of the number, $\text{Re}(z)$ and y is the imaginary part of the number, $\text{Im}(z)$

Powers of i

$i^4 = 1$

$i^0 = 1$

$\times i$

$\times i$

$i^3 = -i$

$i^1 = i$

$\times i$

$\times i$

$i^2 = -1$
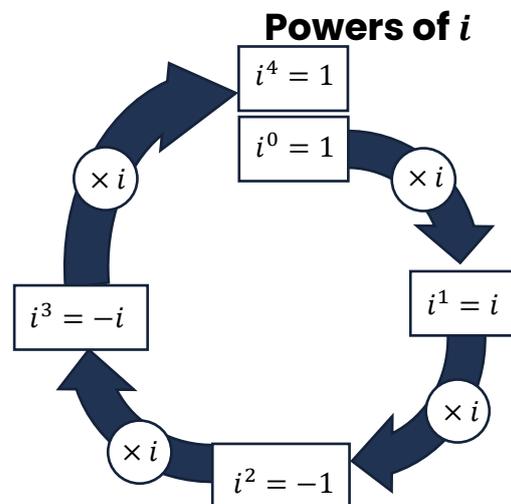
- $i^0 = 1, i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1$ and so on, following the cycle $i, -1, -i, 1$

- $i^{-1} = -i, i^{-2} = -1, i^{-3} = i, i^{-4} = 1$ and so on

Addition and subtraction of complex numbers

To add two complex numbers, the real parts are added and the imaginary parts are added separately. The two results are combined to give the answer

e.g. $(3 - 2i) + (-1 + 5i) = (3 + -1) + (-2 + 5)i = 2 + 3i$

To subtract two complex numbers, the real parts are subtracted in order and the imaginary parts are subtracted in order separately. The two results are combined to give the answer

e.g. $(3 - 2i) - (-1 + 5i) = (3 - -1) + (-2 - 5)i = 4 - 7i$

Calculations with Imaginary and Complex Numbers

The conjugate of a complex number

The conjugate of $z = x + yi$ is $z^* = x - yi$. Only the imaginary part changes its sign.

Multiplication of complex numbers

To multiply two complex numbers, the following procedure is followed

for $(a + bi)(c + di)$

multiply a by c to get ac

multiply $+bi$ by $+di$ to get $+bdi^2$ which is equal to $-bd$ (as $i^2 = -1$)

multiply a by $+di$ to get $+adi$

multiply $+bi$ by c to get $+bci$

combine all of these to get $ac - bd + (ad + bc)i$

An example of multiplication

$(3 + 2i)(2 - 5i) = 6 - 10i^2 - 15i + 4i$

$$= 6 + 10 - 11i$$

$$= 16 - 11i$$

Division of complex numbers

To divide two complex numbers, the following procedure is followed

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- write the division in the form $\dfrac{a+bi}{c+di}$

- multiply both top and bottom of the fraction by $c - di$ i.e. the conjugate of the denominator - the new denominator will be a real number

- separate the result into a real and an imaginary part (this isn't strictly necessary but with complex numbers it is better to be able to see the real and imaginary parts separately).

An example of division

$$(3 + 2i) \div (2 - 5i) = \frac{3 + 2i}{2 - 5i}$$

$$= \frac{3 + 2i}{2 - 5i} \times \frac{2 + 5i}{2 + 5i}$$

$$= \frac{6 + 10i^2 + 15i + 4i}{4 - 25i^2 + 10i - 10i}$$

$$= \frac{6 - 10 + 19i}{4 + 25}$$

$$= \frac{-4 + 19i}{29}$$

$$= -\frac{4}{29} + \frac{19}{29}i$$

A geometrical interpretation of complex numbers

A complex number $z = x + yi$ can be represented geometrically on an Argand diagram.

This is a Cartesian grid with the horizontal axis representing the real part of the number and the vertical axis representing the imaginary part of the number.

The point (x, y) on an Argand diagram represents the complex number $x + yi$.

Complex numbers can also be represented by vectors on an Argand diagram.

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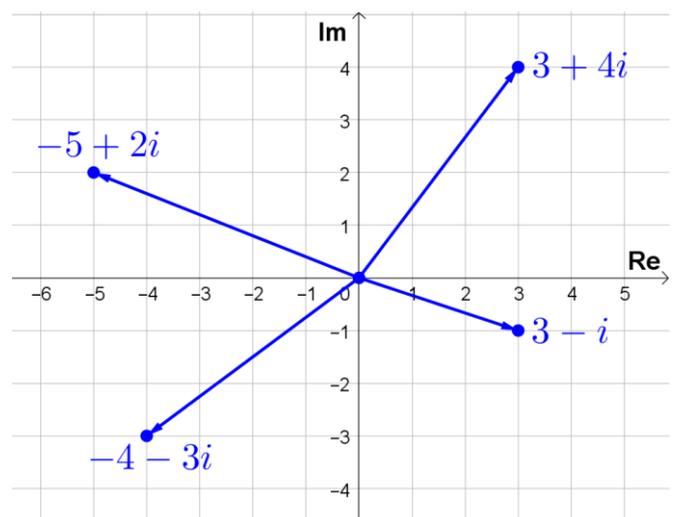
n$^{\text{th}}$ roots and complex numbers

Raising a complex number to a power

You can square a complex number by multiplying it by itself

e.g. $(2 + 3i)^2 = (2 + 3i)(2 + 3i)$

$$= 4 + 9i^2 + 6i + 6i$$

$$= -5 + 12i$$

You can cube a complex number either

by multiplying it by itself and the result by the original complex number	or by using a binomial expansion
e.g. $(2 + 3i)^3 = (2 + 3i)(2 + 3i)^2$ $= (2 + 3i)(-5 + 12i)$ $= -10 + 36i^2 + 24i - 15i$ $= -46 + 9i$	e.g. $(2 + 3i)^3$ $= 2^3 + 3 \times 2^2 \times (3i) + 3 \times 2 \times (3i)^2 + (3i)^3$ $= 8 + 36i + 54i^2 + 27i^3 = 8 + 36i - 54 - 27i$ $= -46 + 9i$ $\left(\text{since } i^2 = -1 \text{ and } i^3 = -i\right)$

You can raise a complex number to the power of 4 either

by squaring it and squaring the result	or by using a binomial expansion
e.g. $(2 + 3i)^4 = (2 + 3i)^2(2 + 3i)^2$ $= (-5 + 12i)^2$ $= 25 + 144i^2 - 60i - 60i$ $= -119 - 120i$	e.g. $(2 + 3i)^4$ $= 2^4 + 4 \times 2^3 \times (3i) + 6 \times 2^2 \times (3i)^2$ $\qquad\qquad +4 \times 2 \times (3i)^3 + (3i)^4$ $= 16 + 96i + 216i^2 + 216i^3 + 81i^4$ $= 16 + 96i - 216 - 216i + 81$ $= -119 - 120i$ $\left(\text{since } i^2 = -1, i^3 = -i \text{ and } i^4 = 1\right)$

Calculate the following (even the really simple ones - there is a reason for this)

a) $(1)^2$

b) $(-1)^2$

c) $(1)^3$

d) $\left(-\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i\right)^3$

e) $\left(-\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i\right)^3$

f) $(1)^4$

g) $(0 + i)^4$

h) $(-1)^4$

i) $(0 - i)^4$

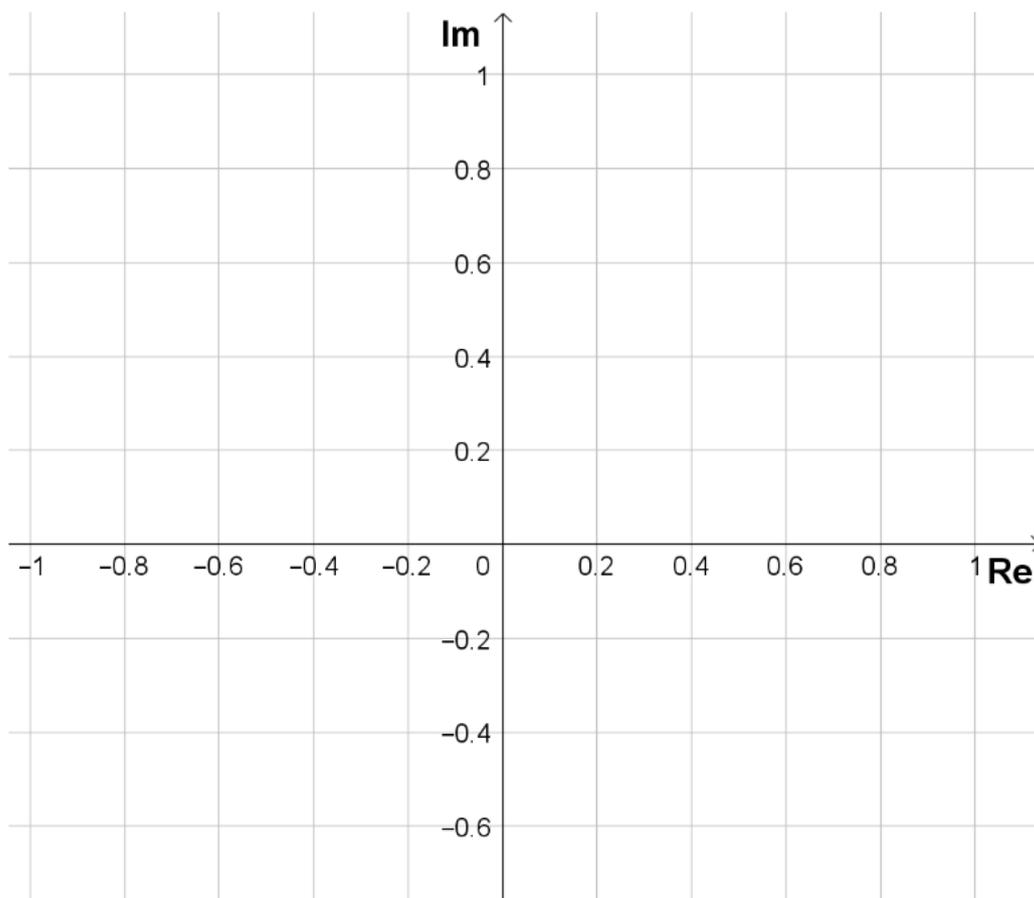What do you notice?

nth roots and complex numbers

Roots of 1 on an Argand diagram

Using the Argand diagram below:

Plot three points, one for each of the cube roots of unity - try to get them close to where they should be on your sketch.

You should be plotting $(1,0)$ for 1, $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ for $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ for $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$
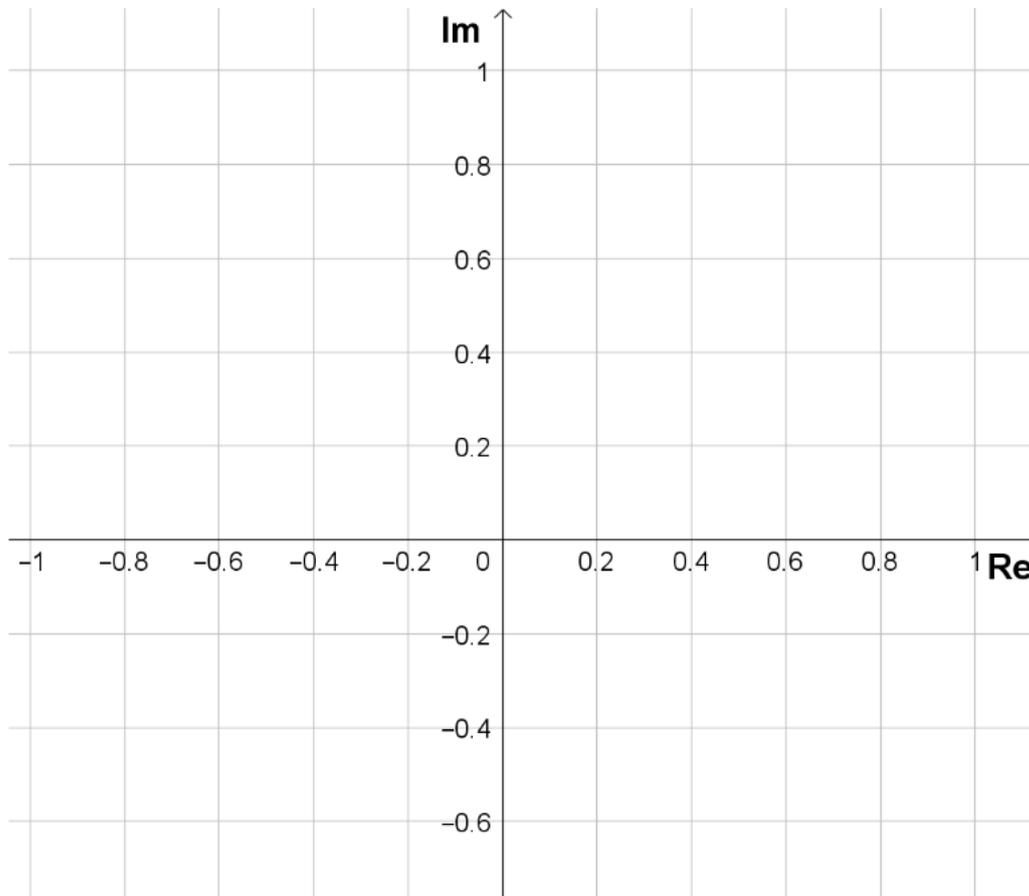

What do you notice about the three points?

Find the distance between each pair of points.

On the Argand diagram below:
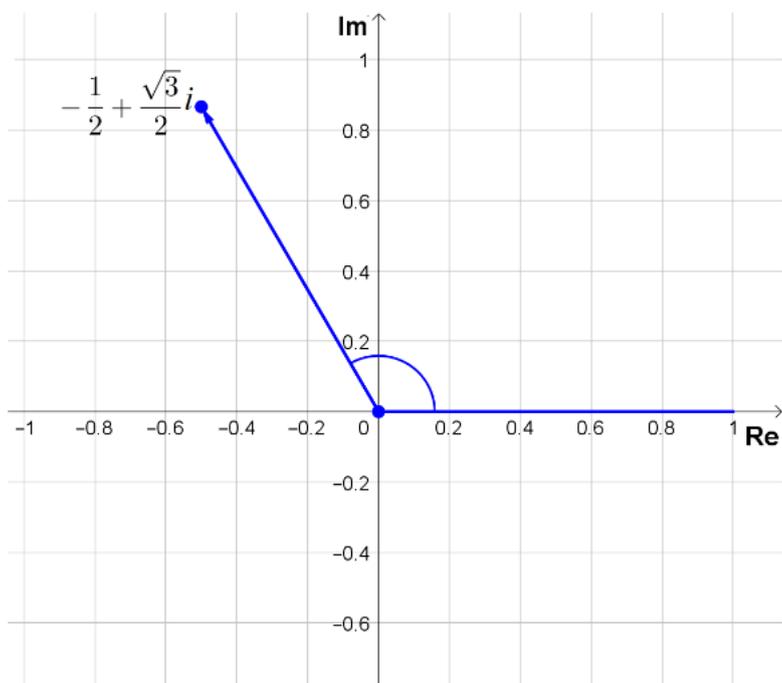
Plot the 4th roots of unity.

This time join the points in order 1 to i to -1 to $-i$.

What do you notice?

Find the distance between each point and the next.

Go back to your diagram showing the cube roots of unity.

Join each point to the centre and then find the angle from each point anticlockwise from the positive Real axis. One of the roots is shown in this diagram:

What is the length of the line drawn from the centre to any of the points?

Do the same for your 4th roots of unity.

You should now have a good idea of how roots of unity appear on an Argand diagram.

The square roots of unity are the solutions to the equation $z^2 - 1 = 0$

The cube roots of unity are the solutions to the equation $z^3 - 1 = 0$

The 4th roots of unity are the solutions to the equation $z^4 - 1 = 0$

Find the 6th roots of unity. Draw them on the Argand diagram below.
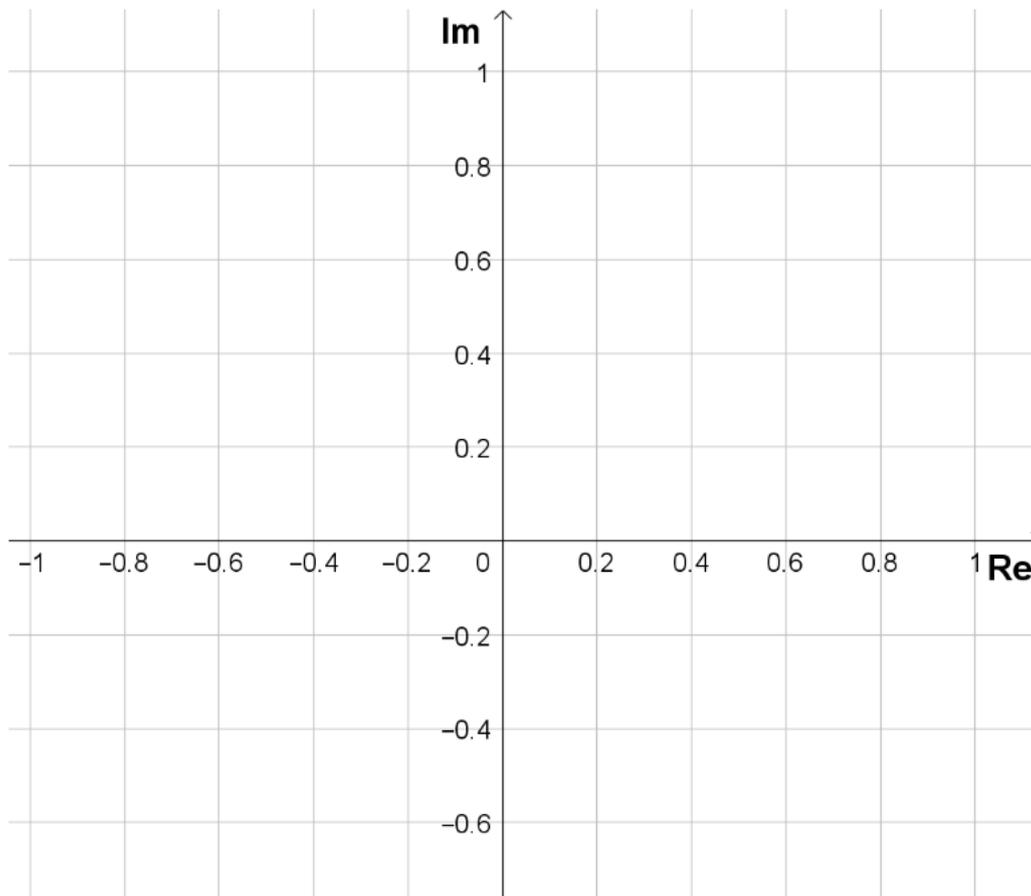
nth roots and complex numbers

The relationship between the roots of unity

Part 1

The cube roots of unity are $1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

Call the second root (anticlockwise from 1) ω. So $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$.

Calculate ω^2. What do you find?

Calculate the value of $1 + \omega + \omega^2$.

Part 2

The 4th roots of unity are $1, i, -1$ and $-i$.

Call the second root (anticlockwise from 1) ω.

Calculate ω^2 and ω^3. What do you find?

Calculate the value of $1 + \omega + \omega^2 + \omega^3$

Part 3

The 6th roots of unity are $1, \frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -1, -\frac{1}{2} - \frac{\sqrt{3}}{2}i$ and $\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

Call the second root (anticlockwise from 1) ω.

Calculate $\omega^2, \omega^3, \omega^4$ and ω^5. What do you find?

Calculate the value of $1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5$.

Do the results you have found work for the 5th roots of unity and the 7th roots of unity?

Find the 5th roots of unity and the 7th roots of unity and see if the relationships you have found work for these cases too.

nᵗʰ roots and complex numbers

Extension – Roots of other complex numbers

Part 1

You have found the cube roots of unity. These are all solutions to the equation $z^3 = 1$.

You can find the cube root of i by solving the equation $z^3 = i$

$z^3 - i = 0$

$z = -i$ is an "obvious" solution $-i \times -i \times -i = (-i \times -i) \times -i = i^2 \times -i = -1 \times -i = i$

By the factor theorem $z + i$ is a factor

Let $(z + i)(z^2 + kz - 1) \equiv z^3 - i$ i.e. attempt to factorise $z^3 - i$

Equating coefficients of z^2 gives $k + i = 0$ so $k = -i$

Hence $(z + i)(z^2 - iz - 1) \equiv z^3 - i$

So the equation $z^3 - i = 0$ can be written as $(z + i)(z^2 - iz - 1) = 0$

The other two cube roots of i come from solving the quadratic equation $z^2 - iz - 1 = 0$

By completing the square $\left(z - \frac{i}{2}\right)^2 - \left(\frac{i}{2}\right)^2 - 1 = 0$

$\left(z - \frac{i}{2}\right)^2 = 1 + \frac{i^2}{4}$

$\left(z - \frac{i}{2}\right)^2 = 1 - \frac{1}{4}$

$\left(z - \frac{i}{2}\right)^2 = \frac{3}{4}$

$z - \frac{i}{2} = \pm \frac{\sqrt{3}}{2}$

$z = \pm \frac{\sqrt{3}}{2} + \frac{1}{2}i$

The cube roots of i are $-i, \frac{\sqrt{3}}{2} + \frac{1}{2}i$ and $-\frac{\sqrt{3}}{2} + \frac{1}{2}i$

Plot them on an Argand diagram.

What do you notice?

How do these numbers relate to the cube roots of unity?

You are now going to find the cube roots of some other numbers:

1. The cube root of -1 by solving $z^3 = -1$
2. The cube root of $-i$ by solving $z^3 = -i$

Part 2

Use what you have discovered to find the 4th roots of i, -1 and $-i$ i.e. solve $z^4 = i$, $z^4 = -1$ and $z^4 = -i$ respectively. Plot each set of roots on an Argand diagram. What do you notice?

Part 3

Use what you have discovered to find the 6th roots of i, -1 and $-i$ i.e. solve $z^6 = i$, $z^6 = -1$ and $z^6 = -i$ respectively. Plot each set of roots on an Argand diagram. What do you notice?

Part 4

Use what you have discovered to find the 8th roots of i, -1 and $-i$ i.e. solve $z^8 = i$, $z^8 = -1$ and $z^8 = -i$ respectively. Plot each set of roots on an Argand diagram. What do you notice?

Matrix Multiplication

Matrix Multiplication

For this workshop you will need to be confident that you can multiply two matrices together.

Here is a reminder.

1. The order of a matrix

$\begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}$ is a 2×2 matrix.

It has 2 rows: $\begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}$

and 2 columns: $\begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}$

The order of the matrix is 2×2. Rows are always written first and columns second.

$\begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & -1 \end{pmatrix}$ is a 2×3 matrix.

This matrix has 2 rows and 3 columns.

The order of the matrix is 2×3

For matrix multiplication, order is important

To be able to multiply two matrices together, the number of **columns** in the first matrix **must** be the same as the number of **rows** in the second. If this is not the case then matrix multiplication can not be done.

If two matrices are of orders that allow for multiplication then they are said to be **conformable** under matrix multiplication.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$$

$$2 \times 3 \quad 3 \times 1$$

The first matrix is order 2×3, the second is order 3×1. The two matrices may be multiplied.

To perform a quick test of conformability, you can write the orders of each matrix down, first matrix before second like this:

$$2 \times 3 \quad 3 \times 1$$

Summary

Conformable…can be multiplied	Non-conformable…cannot be multiplied
$\begin{pmatrix} -2 & 3 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix}$ $2 \times 2 \qquad 2 \times 2$	$\begin{pmatrix} -2 & 3 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & -1 \\ 0 & 3 \end{pmatrix}$ $2 \times 2 \qquad 3 \times 2$

2. The mechanism for matrix multiplication

Matrix multiplication involves multiplying every element in a particular **row** of the **first** matrix by a corresponding element in a particular **column** of the **second** matrix. The results of these multiplications are summed.

The row and column used dictate the position of the element found by doing this in the resulting matrix. As long as you can identify the position of an element in the answer matrix, you know which row and column are multiplied and summed to find it.

Most of the matrices you are going to multiply together are 2×2 or $2 \times n$ matrices.

This diagram shows you how the rows and columns of the two matrices being multiplied relate to the positions of the elements in the answer:

$$\text{row 1} \rightarrow \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -2 & 3 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$$

(col 1, col 2 for the second matrix; row 1, row 2 for the first matrix)

$a_{1,1}$ is the result of multiplying row 1 (of the first matrix) by column 1 (of the second).

$a_{1,1} = (1 \times -2) + (-2 \times 2) = -6$

Find the values of $a_{1,2}$, $a_{2,1}$ and $a_{2,2}$. The answers are on the next page.

$a_{1,2} = (1 \times 3) + (-2 \times 0) = 3$

$a_{2,1} = (-1 \times -2) + (2 \times 2) = 6$

$a_{2,2} = (-1 \times 3) + (2 \times 0) = -3$

$$\begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -2 & 3 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} -6 & 3 \\ 6 & -3 \end{pmatrix}$$

3. The order of the answer matrix

You can use the orders of the matrices being multiplied together to find the order of the resulting matrix.

$$\begin{pmatrix} 2 & 3 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 4 & -1 \\ -3 & 2 & 1 & -2 \end{pmatrix}$$

$$2 \quad \times \quad \boxed{2 \quad \text{conformable} \quad 2} \quad \times \quad 4$$

The first matrix has 2 rows and the second matrix has 4 columns.

The order of the resulting matrix will be 2×4.

Matrix multiplication

Matrix transformation

A linear transformation of a point (x, y) in 2-dimensions is one in which the x and y coordinates of the new point are calculated by a linear combination of the x and y coordinates of the original point.

Original point (object)		Transformed point (image)
(x, y)	$\longrightarrow$	(x', y')

New x coordinate: $x' = ax + by$

New y coordinate: $y' = cx + dy$

Example

Reflection in the line $y = x$

The new x coordinate will be equal to the old y coordinate and

the new y coordinate will be equal to the old x coordinate so
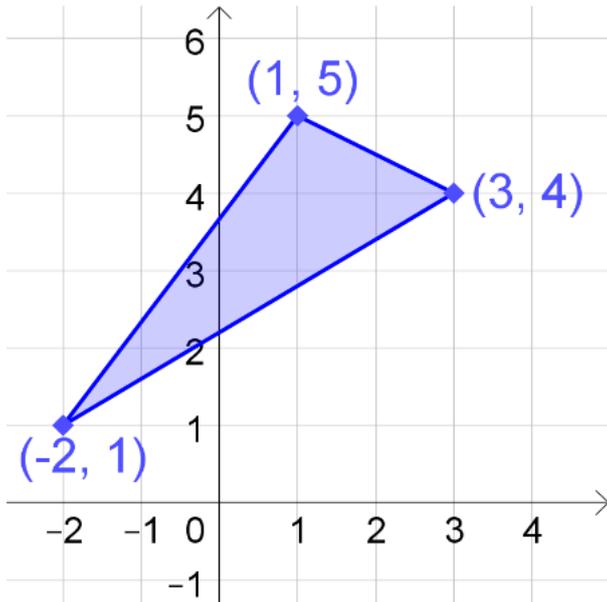
$x' = 0x + 1y$

$y' = 1x + 0y$

Matrix multiplication and transformation

Matrix multiplication can be used to apply linear transformations.

A reflection in the line $y = x$ can be achieved by pre-multiplying by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \longrightarrow \qquad \begin{array}{l} x' = 0x + 1y \\ y' = 1x + 0y \end{array}$$

Transforming a shape

This triangle has vertices at $(-2,1)$, $(3,4)$ and $(1,5)$.
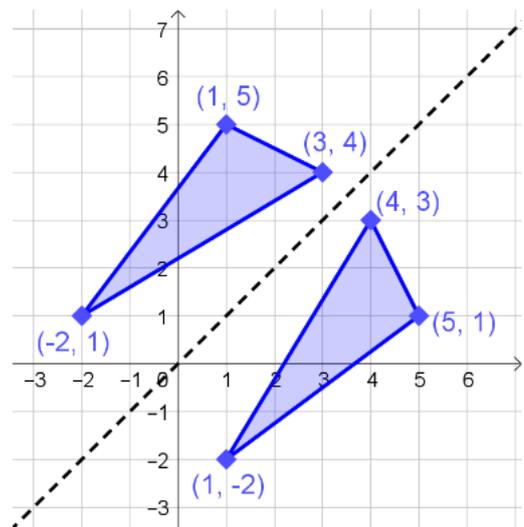
These vertices are represented in a matrix as

$$\begin{pmatrix} -2 & 3 & 1 \\ 1 & 4 & 5 \end{pmatrix}$$

The triangle can be reflected in the line $y = x$ by the matrix multiplication

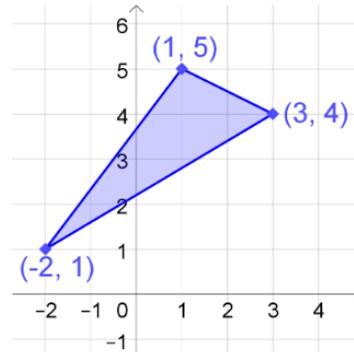$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -2 & 3 & 1 \\ 1 & 4 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -2 & 3 & 1 \\ 1 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 5 \\ -2 & 3 & 1 \end{pmatrix}$$

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Activity 1

Each of the following matrices represents a linear transformation.

Apply each transformation to this triangle:

a) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

b) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

c) $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

What transformation is achieved by each matrix multiplication?

Matrix multiplication

Some things never change

The point $(7,7)$ on the original plane transforms to $(7,7)$ for the transformation represented by the matrix $\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$ since $\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} 7 \\ 7 \end{pmatrix} = \begin{pmatrix} (2 \times 7) + (-1 \times 7) \\ (3 \times 7) + (-2 \times 7) \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \end{pmatrix}$.

Find some other points that don't change position under the transformation represented by $\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$.

There is one "obvious" point but are there any others?

Can you think of a way to set up an equation to find the points that don't change position?

Is there anything you can say about all of the points you have found?

Matrix multiplication

Lines of invariant points

Find any invariant points or any lines of invariant points for the following matrix transformations:

a) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

b) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

c) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

d) $\begin{pmatrix} 3 & 2 \\ -4 & -3 \end{pmatrix}$

e) $\begin{pmatrix} -5 & 2 \\ 0 & 1 \end{pmatrix}$

f) $\begin{pmatrix} 6 & 1 \\ 5 & 2 \end{pmatrix}$

Matrix multiplication

Find the image of each of the following points under the transformation $\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$

a) $(1,3)$

b) $(-3,9)$

c) $(5,15)$

d) $(-100,-300)$

What do you notice?

Activity 4 - Challenge

Find the invariant lines and the lines of invariant points for the following matrix transformations:

a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

b) $\begin{pmatrix} 3 & 2 \\ -4 & -3 \end{pmatrix}$

c) $\begin{pmatrix} 2 & 1 \\ -4 & -3 \end{pmatrix}$

d) $\begin{pmatrix} -3 & 1 \\ -7 & 5 \end{pmatrix}$

Hint: For both invariant lines and lines of invariant points, $M\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$.

For lines of invariant points $\lambda = 1$