Two Hours

UNIVERSITY OF MANCHESTER

PROBABILITY 1

23 January 2019

1400 - 1600

Answer all 6 questions

Electronic calculators may be used, provided that they cannot store text.

1.

i) In the context of sets, define union, intersection and complement. [**3 marks**]

ii) Using a Venn diagram, show that

$$(A \cup B)^c = A^c \cap B^c \,.$$ [**3 marks**]

iii) Using a Venn diagram, show that

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)\,.$$ [**3 marks**]

iv) Define what it means for events A and B to be independent. [**3 marks**]

v) Show that, if A and B are independent, then so are A^c and B^c. [**3 marks**]

[**Total: 15 marks**]

2. You repeatedly roll a biased die, where, on any given roll, the probability of each outcome is $(p_i : i = 1, ..., 6)$. Let T be the first roll on which the die does *not* show a 6 and let X be the number shown on roll T.

i) Give the probability distribution of T. [**4 marks**]

ii) What is the mean of T? [**4 marks**]

iii) Derive the joint probability distribution of T and X. That is, find

$$\mathbb{P}(T = t, X = x) \qquad \text{for} \quad t \in \mathbb{N} \quad \text{and} \quad x = 1, ..., 5.$$ [**4 marks**]

iv) Are X and T independent? Justify your answer. [**3 marks**]

[**Total: 15 marks**]

3. (a) i) Give the definition of the expectation of a discrete random variable T . **[3 marks]**

ii) Assuming that T takes values in $\mathbb{N}$, show that

$$\mathbb{E}[T] = \sum_{t=0}^{\infty} \mathbb{P}(T > t)$$

[Hint: consider the sum, $\sum_{t=0}^{\infty} \mathbb{I}[T > t]$.] **[4 marks]**

(b) In a tutorial with 8 students, the lecturer asks each student the day of the week of their birthday.

i) Determine the probability that, after asking 4 students, she has not found two students with the same birthday? **[4 marks]**

ii) Using your answer to part a), calculate, correct to two decimal places, the expected number of students the lecturer must ask before she finds two students with the same birthday. **[4 marks]**

[Total: 15 marks]

4. X_1 and X_2 are independent and uniformly distributed on the interval $[0, 1]$. Their common probability density function is given by

$$f(x) = \begin{cases} 1, & \text{for } 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

i) Calculate the mean and variance of X_1 . **[5 marks]**

Let $Y_1 = \min\{X_1, X_2\}$ and $Y_2 = \max\{X_1, X_2\}$.

ii) Show that $\mathbb{P}(Y_1 \ge y) = (1 - y)^2$. **[3 marks]**

iii) What is the probability density function of Y_1 . **[2 marks]**

iv) Calculate the expectation of Y_1 . **[3 marks]**

v) Without further calculation, give the expectation of Y_2 . **[2 marks]**

[Total: 15 marks]

5. Let $X_1, ...X_n$ be independent discrete random variables with the same distribution. The expection of each X_i is μ and the standard deviation of each X_i is σ . Let

$$S_n = \sum_{i=1}^{n} X_i \, .$$

i) Give the definition of $Var(X)$, the variance a discrete random variable X ; and give the definition of the standard deviation of X . **[3 marks]**

ii) Show that

$$Var(aX + b) = a^2 Var(X)$$ **[3 marks]**

iii) Show that, for X_1 and X_2 introduced above,

$$Var(X_1 + X_2) = Var(X_1) + Var(X_2)$$ **[3 marks]**

iv) Show that

$$Var\Big(\frac{S_n - \mu n}{\sigma\sqrt{n}}\Big) = 1$$ **[3 marks]**

v) State the Central Limit Theorem. **[3 marks]**

[Total: 15 marks]

6. (a) The number of phone calls per hour, made in a certain village, is Poisson distributed with mean 2. The numbers of calls made in non-overlapping intervals are independent.

i) Determine the probability of no phone calls between 1 pm and 5 pm? **[3 marks]**

ii) Determine the probability of two or more phone calls between 1 pm and 5 pm? **[4 marks]**

[Hint : For Poisson random variables if $X \sim Po(\lambda)$, $Y \sim Po(\mu)$ and X and Y are independent then $X + Y \sim Po(\lambda + \mu)$.]

(b) The length of each call is well approximated by a normal distribution with mean 35 and variance 10 . Given this approximation:

i) Determine the proportion of calls of duration greater than 45 minutes; **[4 marks]**

ii) Determine the proportion of calls of duration between 10 and 15 minutes. **[4 marks]**

[Total: 15 marks]

END OF EXAMINATION PAPER