<u>Three hours</u>

A formula sheet is provided at the end of the examination

THE UNIVERSITY OF MANCHESTER

CALCULUS AND VECTORS A

25 January 2019

14:00 – 17:00

Answer **ALL FIVE** questions.

University approved calculators may be used.

1.

(a) Show that $z = \pm \mathrm{i}$ both satisfy the equation

$$z^4 - 2z^3 + 6z^2 - 2z + 5 = 0. \tag{1}$$

[2 marks]

(b) Hence, or otherwise, find all solutions of the equation (1) and mark them on a sketch of the complex plane.

[5 marks]

(c) Sketch the function $f(x) = 4\pi^2 x - x^3$ and find the area bounded by the lines $y = f(x)$, $y = 0$, $x = 0$ and $x = 2\pi$.

[7 marks]

(d) Sketch the function $r = f(\theta)$, where the function f is defined in (c) and (r, θ) are plane-polar coordinates. Find the area enclosed by the curve $r = f(\theta)$ over the range $\theta \in [0, 2\pi]$.

[6 marks]

Total marks: 20

2.

(a) Use Euler's formula to prove that

$$\cos(A + B) = \cos A \cos B - \sin A \sin B.$$

Hence, or otherwise, prove also that

$$\cosh(A + B) = \cosh A \cosh B + \sinh A \sinh B.$$

[4 marks]

(b) Find the first three non-zero terms in the Taylor Series expansions for $\sin x$ and $\cos x$ about $x = 0$.

[4 marks]

(c) On the same axes, sketch the functions $\cosh x$ and $\cosh^{-1} x$ and specify the domain and range of each function.

[3 marks]

(d) Use the results from (a) and (b) and the fundamental definition of the derivative to show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\cosh x\right] = \sinh x.$$

[3 marks]

(e) Find $\frac{\mathrm{d}}{\mathrm{d}x}\left[\cosh^{-1} x\right]$, assuming $x > 1$.

[3 marks]

(f) Hence, or otherwise, find

$$\int \frac{1}{\sqrt{81x^2 + 18x}}\,\mathrm{d}x,$$

assuming $x > 0$.

[3 marks]

Total marks: 20

3.

(a) The line L_1 passes through the points A, B and the line L_2 passes through the points C, D. The coordinates of the points are

$$A(1,0,0), \quad B(0,1,1), \quad C(0,0,1), \quad D(1,1,1+\alpha)$$

where $\alpha \in \mathbb{R}$.

(i) Determine the parametric equations of L_1 and of L_2 (the latter will involve α)

(ii) For which value α_0 of α do the lines intersect? Find the point of intersection in this case.

(iii) When $\alpha = \alpha_0$, the two lines lie in a common plane. Find an equation for this plane.

(iv) For general values of α, show that the perpendicular distance between the lines is equal to

$$d = \frac{|\alpha - \alpha_0|}{\sqrt{b + c\alpha^2}}$$

where the values of b and c are to be determined.

(v) Find the value of α that maximises that distance (you do not need to check it is a maximum).

(vi) Sketch the graph of the function d from part (iv) as a function of α, making clear any turning points and asymptotes.

[17 marks]

(b) Let $g(x, y) = \sin(x)\cos(x - y)$.

(i) Find all critical points of g;

(ii) calculate the Taylor series to order 2 (i.e., ignoring terms of degree 3 or more) of this function about the point $(x_0, y_0) = (0, \pi/2)$.

[5 marks]

Total marks: 22

4.

(a) Let D be the unit disk in the plane, centred at the origin. By changing to polar coordinates, evaluate the integral

$$\iint_D \frac{1}{\sqrt{x^2+y^2}}\,\mathrm{d}x\,\mathrm{d}y.$$

[4 marks]

(b) Consider the double integral

$$I = \int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} x^2 \sin(xy)\,\mathrm{d}x\,\mathrm{d}y.$$

(i) Sketch the region R of integration, showing clearly the relevant information.

(ii) Express the integral I with the order of integration reversed.

(iii) Evaluate the integral.

[6 marks]

(c) Green's theorem states that, for functions P and Q of x and y,

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y \;=\; \oint_C P\,\mathrm{d}x + Q\,\mathrm{d}y.$$

(i) State what C and D represent, and how they are related.

(ii) Sketch the region Ω given by $0 \le x \le 2$ and $0 \le y \le 2$, and use Green's theorem to evaluate the double integral

$$\iint_\Omega (3x^2 + 2xy)\,\mathrm{d}x\,\mathrm{d}y.$$

[6 marks]

Total marks: 16

5.

(a) Suppose u, v and z are related by the equation

$$z^2 - 2uv = \mathrm{e}^{u+v}.$$

Find $\left.\frac{\partial z}{\partial u}\right|_v$, $\left.\frac{\partial u}{\partial v}\right|_z$ and $\left.\frac{\partial v}{\partial z}\right|_u$ and determine the product

$$\left.\frac{\partial z}{\partial u}\right|_v \left.\frac{\partial u}{\partial v}\right|_z \left.\frac{\partial v}{\partial z}\right|_u$$

simplifying your answer.

[6 marks]

(b) Let $f(x, y, z) = xz + \cos(xy)$. First find the gradient vector field $\mathbf{u} = \vec{\nabla} f$. Now calculate both $\vec{\nabla} \cdot \mathbf{u}$ (divergence) and $\vec{\nabla} \times \mathbf{u}$ (curl) for that vector field.

[4 marks]

(c) Consider the function $f(x, y) = x^3 + 6xy + y^3$ for $(x, y) \in \mathbb{R}^2$.

(i) Find any critical points and identify whether each is a local maximum, a local minimum or a saddle point.

(ii) Provide an approximate sketch of the contours of the function $f(x, y)$.

(iii) If the values of x and y are constrained to lie on the straight line $y + x = 1$, use the Lagrange multiplier method to find the value of (x, y) where $f(x, y)$ has a turning point on the line.

(iv) Find the value of the function f at the turning point found in (iii), and by considering nearby values, or otherwise, determine whether this is a local minimum or a local maximum of f on the line in question.

[12 marks]

Total marks: 22

***Some Useful Formulae*:**

$$f(x) = \sum_{n=0}^{\infty} \frac{\partial_x^n f(x_0)}{n!} (x - x_0)^n$$

$$g(x,y) = \sum_{k=0}^{\infty} \sum_{\{m+n=k\}} \frac{\partial_x^m \partial_y^n g(x_0, y_0)}{m!\, n!} (x - x_0)^m (y - y_0)^n$$

$$\cosh^2 v = 1 + \sinh^2 v$$

$$\cosh^2 v = \tfrac{1}{2}\big(1 + \cosh(2v)\big)$$

$$\sinh(2v) = 2 \sinh v \cosh v$$

$$\cos^2 t = \tfrac{1}{2}\big(1 + \cos(2t)\big)$$

$$\sin(2t) = 2 \sin t \cos t$$

$$\vec{\nabla} \cdot (u, v, w) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$\vec{\nabla} \times (u, v, w) = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$\mathrm{d}x\,\mathrm{d}y = \left| \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix} \right| \mathrm{d}r\,\mathrm{d}\theta$$

$f(x)$	$\int f(x)\,\mathrm{d}x$
$\frac{1}{1+x^2}$	$\tan^{-1} x$
$\frac{1}{1-x^2}$, $\quad \lvert x\rvert < 1$	$\tanh^{-1} x$
$\frac{1}{1-x^2}$, $\quad \lvert x\rvert > 1$	$\coth^{-1} x$
$\frac{1}{1-x^2}$, $\quad \lvert x\rvert \neq 1$	$\frac{1}{2} \ln \left\lvert \frac{x+1}{x-1} \right\rvert$
$\frac{1}{\sqrt{1-x^2}}$, $\quad \lvert x\rvert < 1$	$\sin^{-1} x$
$\frac{1}{\sqrt{1+x^2}}$	$\sinh^{-1} x$
$\frac{1}{\sqrt{x^2-1}}$, $\quad \lvert x\rvert > 1$	$\frac{x}{\lvert x\rvert} \cosh^{-1} \lvert x\rvert$
$\frac{1}{\sqrt{x^2 \pm 1}}$	$\ln \lvert x + \sqrt{x^2 \pm 1} \rvert$

END OF EXAMINATION PAPER