Two and a half hours

THE UNIVERSITY OF MANCHESTER

FOUNDATIONS OF PURE MATHEMATICS B

14 January 2019

14.00 – 16.30

Answer **ALL** EIGHT questions in Section A (40 marks in total).
Answer FOUR of the SIX questions in Section B (60 marks in total).
(If more than four questions are attempted in Section B, credit will be given for the best four answers.)

Electronic calculators may be used in accordance with the University regulations

---

SECTION A

Answer **ALL** of the EIGHT questions

A1.

(a) Write one sentence to describe the meaning of each of the following statements:

(i) $\forall x \in \{2k : k \in \mathbb{Z}\}, \exists a, b \in \mathbb{Z}, x = (2a+1) + (2b+1)$.

(ii) $\forall x \in \mathbb{Z}, \exists a, b \in \{2k+1 : k \in \mathbb{Z}\}, (2 \mid x) \wedge (x = a + b)$.

(b) Write down the negation of statement (i).

(c) Give a counterexample to statement (ii).

[5 marks]

A2. Let p, q and r be propositions.

(a) Use truth tables to determine whether "$(p \Rightarrow q) \Rightarrow r$" and "$p \Rightarrow (q \Rightarrow r)$" are logically equivalent.

(b) What is the *converse* of $p \Rightarrow q$?

(c) What is the *contrapositive* of $p \Rightarrow q$?

[5 marks]

A3. Use induction to prove that $2^{2n} - 1$ is divisible by 3 for every natural number n.

[5 marks]

A4. Let $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$, $C = \{1, 4\}$, $D = \{2, 5\}$.

(a) Compute $X = (A \times B) \setminus (C \times D)$ and $Y = (A \setminus C) \times (B \setminus D)$.

(b) Hence find $X \setminus Y$ and $Y \setminus X$.

[4 marks]

A5. Let A be a set and n a natural number. Give the definitions of each of the following properties:

(a) $|A| = n$.

(b) A is countable.

[4 marks]

A6.

(a) Use the Euclidean algorithm to find $\gcd(2019, 399)$

(b) Find all integer solutions to the following linear congruences, or explain why there are none:

(i) $399x \equiv 7 \bmod 2019$.

(ii) $399x \equiv 60 \bmod 2019$.

[7 marks]

A7. Let R be a relation on a set A.

(a) Define what it means for R to be reflexive.

(b) Define what it means for R to be symmetric.

(c) Define what it means for R to be transitive.

[5 marks]

A8. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and let f and g be the permutations of A defined by $f = (12)(23)(34)(56)(78)(56)(59)$ and $g = (1234)(56789)$.

(a) Write the permutation f as a product of disjoint cycles.

(b) Write down the inverse of f in cycle notation.

(c) Find $f^{-1} \circ g \circ f$, expressing the result in both two-line notation and cycle notation.

[5 marks]

SECTION B

Answer **FOUR** of the SIX questions.

B9.

(a) Let $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ be the functions given by $f(x) = |x| + 1$ and $g(x) = |x| - 1$. What is the image of the composite function $g \circ f$?

[2 marks]

(b) Let $h : \mathbb{C} \to \mathbb{R}$ be the function defined by $h(a + bi) = a$. Prove or disprove each of the following statements:

(i) h is injective. (ii) h is surjective. (iii) h has an inverse.

[5 marks]

(c) Let A, B and C be sets and let $f : A \to B$ and $g : B \to C$ be functions.

(i) Prove that if f and g are both 1-1, then $g \circ f$ is 1-1.
(ii) Prove that if f and g are both onto, then $g \circ f$ is onto.

[8 marks]

B10.

(a) Let A and B be sets and $f : A \to B$ a function.

(i) Explain what it means for f to be injective.
(ii) Explain what it means for f to be surjective.
(iii) Suppose that $|A| = |B| = n$. Show that there is a bijection $h : A \to B$. (You may use without proof the fact that a composition of bijections is again a bijection.)
(iv) Give an example of finite sets A and B with $|A| = |B|$ and a non-bijective function $f : A \to B$.

[8 marks]

(b) State (without proof) the principle of inclusion-exclusion.

[3 marks]

(c) Let A be a set, and n a natural number.

(i) Suppose $|A| = n$. State (without proof) the cardinality of $\mathcal{P}(A)$, and hence find the cardinality of $\mathcal{P}(A) \cup \{\emptyset, A \times A\}$.
(ii) What is the cardinality of $\mathcal{P}(\emptyset) \cup \{\emptyset, \emptyset \times \emptyset\}$?

[4 marks]

B11.

(a) Let A be a non-empty set and R an equivalence relation on A.

(i) For $a \in A$, define the equivalence class R_a of a.

(ii) If $a, b \in A$ and aRb, prove that $R_a = R_b$.

(iii) If $a, b \in A$ and $a \not R b$, prove that $R_a \cap R_b = \emptyset$.

[7 marks]

(b) Consider the relation R defined on $\mathbb{Z}$ as follows: for $a, b \in \mathbb{R}$,

$$a \, R \, b \Leftrightarrow 5 \mid 2a + 3b.$$

(i) Show that R is an equivalence relation.

(ii) Find the equivalence class R_1.

[8 marks]

B12.

(a) Let $*$ be a binary operation on a non-empty set G.

(i) Let $e \in G$. Define what it means for e to be an identity element with respect to $*$.

(ii) Define what it means for $(G, *)$ to be a group.

[5 marks]

(b) Consider the binary operation $*$ on $\mathbb{N} \times \mathbb{N}$ given by $(a, b) * (c, d) = (ac, bd)$ for all $(a, b), (c, d) \in \mathbb{N}$. Prove or disprove each of the following statements:

(i) $*$ is commutative.

(ii) $*$ is associative.

(iii) $(\mathbb{N} \times \mathbb{N}, *)$ is a group.

[6 marks]

(c) Write down the multiplication tables of $(\mathbb{Z}_4, \oplus)$ and $(\mathbb{Z}_4, \odot)$, where $\oplus$ and $\odot$ denote addition modulo and multiplication modulo 4.

[4 marks]

B13.

(a) Let $a, b, c, d \in \mathbb{Z}$ and $n \in \mathbb{N}$.

(i) Define what is meant by $a \equiv b \mod n$.

(ii) Suppose $a \equiv b \mod n$ and $c \equiv d \mod n$. Show that $ac \equiv bd \mod n$.

(iii) Give an example in which $ad \equiv bd \mod n$, but $a \not\equiv b \mod n$.

[7 marks]

(b) State (without proof) Fermat's little theorem.

[3 marks]

(c) (i) Find $x \in \{0, \ldots, 10\}$ such that $x \equiv 2^{555} \mod 11$.

(ii) Find $a \in \mathbb{Z}$ such that $a^6 \not\equiv a \mod 6$.

[5 marks]

B14.

(a) Let $a, b, s, t \in \mathbb{N}$.

(i) Explain what $a \mid b$ means and give the definition of $\gcd(a, b)$.

(ii) Show that if $as + bt = 1$, then $\gcd(a, b) = 1$.

[5 marks]

(b) State (without proof) the fundamental theorem of arithmetic.

[3 marks]

(c) Prove that $\sqrt{7}$ is irrational.
(You may assume that if p is a prime then $\forall a, b \in \mathbb{Z}$, $p \mid ab \Rightarrow p \mid a$ or $p \mid b$.)

[7 marks]

END OF EXAMINATION PAPER