Three hours

THE UNIVERSITY OF MANCHESTER

FOUNDATIONS OF PURE MATHEMATICS A

14 January 2019

14.00 – 17.00

Answer ALL TEN questions

Electronic calculators may be used in accordance with the University regulations

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SECTION A

Answer **ALL FIVE** questions

A1.

(i) Write down the negation of the statement

$$\forall x \in \mathbb{R},\ \exists y \in \mathbb{R},\ x > 3y.$$

State whether the statement or its negation is true. Explain your answer.

(ii) Write down the contrapositive of the statement

'For any $n \in \mathbb{Z}$, if 5 does not divide n^2, then 5 does not divide n.'

Prove the statement is true. [5 marks]

A2.

(i) Let A and B be subsets of a universal set U. Prove that $(A \cup B)^c = A^c \cap B^c$.

(ii) Let $A = \{1, 2, 3\}$ and let the function $f : A \times A \to \mathbb{Z}$ be defined by $f((a, b)) = a - b$. Write down $\operatorname{Im} f$, listing all the elements.

Is f injective? Explain your answer.

(iii) State the Pigeonhole Principle. [5 marks]

A3.

(i) Use the method of successive squaring to find the remainder of 2^{65} when divided by 100.

(ii) Hence or otherwise, find the last two decimal digits of 798^{65}.

(iii) You are given that $n \in \mathbb{N}$ is such that 2^n leaves remainder 2 when divided by 100. Prove by contradiction that $n = 1$. [5 marks]

A4. Let $\phi\colon \mathbb{N} \to \mathbb{N}$ be Euler's phi-function. Let p, q be prime numbers such that $p \neq q$, and let $k \in \mathbb{N}$.

(i) Write down a formula for $\dfrac{\phi((pq)^k)}{(pq)^k}$.

(ii) Hence or otherwise, show that $10^{-k}\phi(10^k) = 0.4$.

(iii) Is it possible for $\phi((pq)^k)$ to be a prime number? Explain your answer. [5 marks]

A5. The permutations $\rho, \sigma, \tau \in S_8$ are given by

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 1 \end{pmatrix},\ \sigma = (1,2) \circ (2,3) \circ (3,4) \circ (4,5) \circ (2,3) \circ (1,2),\ \tau = \rho \circ \sigma \circ \rho^{-1}.$$

(i) By writing τ as a product of disjoint cycles, show that τ is a cycle of length 3. State the order of τ.

(ii) How many cycles of length 3 are there in S_8? Explain your answer briefly. (Your answer may contain binomial coefficients or factorials, and you do not have to calculate their numerical values.)

[5 marks]

SECTION B

Answer **ALL FIVE** questions

B6.

(i) Let $P(n)$ be a predicate. Describe the method of simple induction used to prove that $P(n)$ is true for all $n \in \mathbb{N}$. [2 marks]

(ii) Use simple induction to prove that $2^{n-1} \leq n!$ for all $n \in \mathbb{N}$. [4 marks]

(iii) Let $f : \mathbb{R} \to \mathbb{R}\backslash\{0\}$ and $g : \mathbb{R}\backslash\{0\} \to \mathbb{R}^+$ be defined by

$$f(x) = e^x, \text{ for all } x \in \mathbb{R}, \qquad g(x) = \frac{1}{x^2} \text{ for all } x \in \mathbb{R}\backslash\{0\}.$$

Write down $g \circ f$, stating the domain and codomain. Is $g \circ f$ surjective? Explain your answer. [4 marks]

B7.

(i) Define what it means to say that a set is denumerable. Prove that the set of even integers is denumerable. [3 marks]

(ii) Define what it means for two sets to be equipotent.

Let A, B and C be sets. Prove that if A and B are equipotent and B and C are equipotent, then A and C are equipotent. (You should state, without proof, any properties of functions you use.)

Prove that the open intervals $(0, 1)$ and $(0, 10)$, subsets of the set of real numbers, are equipotent. [5 marks]

(iii) Write the repeating decimal $0.3\overline{457}$ as $\frac{m}{n}$ where m, n are integers. [2 marks]

B8. Let $(x_0, y_0) \in \mathbb{Z}^2$ be an integer solution of the equation $ax + by = c$ where a, b, c are integers.

(i) Assuming that $a \neq 0$ and $b \neq 0$, prove that if $(x, y) \in \mathbb{Z}^2$ is a solution of this equation, then

$$(x, y) = \big(x_0 - \frac{b}{\gcd(a, b)}t,\ y_0 + \frac{a}{\gcd(a, b)}t\big)$$

for some $t \in \mathbb{Z}$. Results from the course used in the proof must be stated, but you do not have to prove them. [6 marks]

(ii) Now assume that $a = 0$ and $b \neq 0$. Is it still true that if $(x, y) \in \mathbb{Z}^2$ is a solution of the equation $ax + by = c$, then (x, y) is given by the formula from (i)? Justify your answer. [4 marks]

B9. Let the relation $\sim$ be defined on the set $\mathbb{Z}$ as follows: for $a, b \in \mathbb{Z}$, $a \sim b$ if and only if $4 \,|\, (7a^3 + b^3)$.

(i) Prove that $\sim$ is an equivalence relation. [5 marks]

(ii) Show that the equivalence class of the integer 0 is the set of even integers. [3 marks]

(iii) Write down, without proof, one other equivalence class induced by the relation $\sim$. [2 marks]

B10.

(i) Give the definition of a prime number.

Deduce from the definition that every prime number p has the following property: for all integers a, a is divisible by p or a is coprime to p.

Is there any non-prime number $p \in \mathbb{N}$ which has this property? Explain your answer.

[5 marks]

(ii) Let P be the set of all prime numbers. Prove: $\exists S \subseteq P$, $1 \leqslant |S| \leqslant 2^{99}$, $\left(\prod_{p \in S} p\right) \equiv 1 \mod 2^{99}$.

[5 marks]

END OF EXAMINATION PAPER