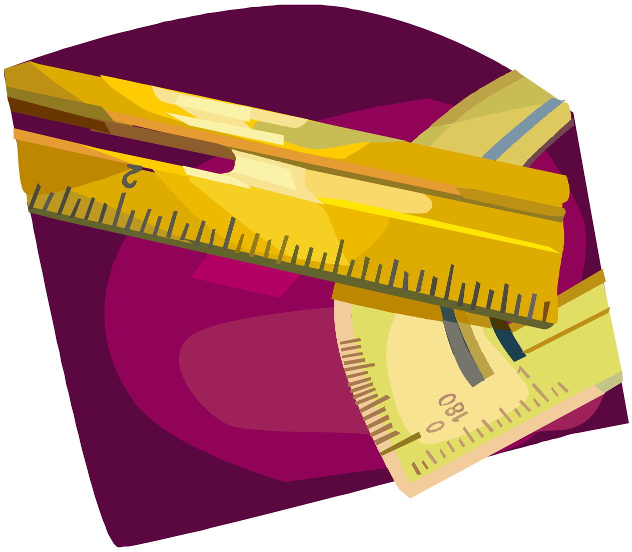
GCSE MATHEMATICS HELP BOOKLET
School of Social Sciences



This is a guide to ECON10061 (introductory Mathematics)
Whether this guide applies to you or not please read the explanation inside

Acknowledgements

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PREFACE

This booklet is intended to help those who have done their GCSE Maths a while ago and may have forgotten some of the material they learnt. It will help them revise the basic tools (covered in the GCSE Maths syllabus), which are needed to take the Introductory Maths course of the School of Economics Studies of the University of Manchester.

The tools of mathematics are widely used in areas such as economics, business, accounting and finance, and being confident with these tools can only help students to successfully understand and study these subjects.

We hope that you will work through this booklet, to give yourself a good start to your studies when come to Manchester.

The booklet is divided into two main topics: Algebra and Numbers, which are subdivided into different subtopics. Within each these topics we have tried to go from the easiest examples to more difficult ones. However, as far as the topics themselves are concerned you are free to start from Topic 1 or 2. At the end of the booklet you will find answers to all the exercises.

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Topic 1 ALGEBRA

1. Manipulation

1.1. Simplification of Expression

This concept is best understood by trying an example. Let us assume you are asked to simplify the following expression:

$$3a-4b+2a+5b-a-b \qquad (1)$$

What you need to remember is to gather terms that are alike together. In other words, you can only add the as together and the bs together (but you cannot add as to bs). Another important thing you need to remember is, when adding up terms that are alike, you need to take into account the + or - sign in front of each term.

Now let us try to simplify example (1) above.

$$\left.\begin{array}{l} 3a+2a-a=5a-a=4a \\ -4b+5b-b=-5b+5b=0 \end{array}\right\} \textit{this implies that } 3a-4b+2a+5b-a-b=4a+0=\boxed{4a}$$

Let us try another example this time using xs and ys instead of as and bs. Assume we were asked to simplify the following expression:

$$-3x+y-7y+x-y-5x+2y \qquad (2)$$

Following the same steps as before we get:

$$\left.\begin{array}{l} -3x+x-5x=-8x+x=-7x \\ +y-7y-y+2y=3y-8y=3y-8y=-5y \end{array}\right\} \textit{this implies that } -3x+y-7y+x-y-5x+2y=\boxed{-7x-5y}$$

Now let us try something a bit more complicated. Let us simplify the following expression:

$$-x^3-2x^2-1+x+2x^3+x^2-6-4x+2 \qquad (3)$$

In this example we have 4 different terms ($x^3s,\ x^2s,\ xs,$ *and the numbers*). Again we need to put the like terms together, as below:

$$\left.\begin{array}{l} -x^3+2x^3=x^3 \\ -2x^2+x^2=-x^2 \\ +x-4x=-3x \\ -1-6+2=-5 \end{array}\right\} \textit{this implies that } -x^3-2x^2-1+x+2x^3+x^2-6-4x+2=\boxed{x^3-x^2-3x-5}$$

As these examples show, simplifying an expression just means rewriting it in such a way that no term is repeated twice or more in the final version. The importance of simplification will become clear in the section reserved to solving linear equations. Now here are few examples for you to work through.

Exercise 1.1.1

Simplify the following expressions:

(a) $2x-4+5x+6$

(b) $2x^2+3xy-4y^2+2xy-3x^2+y^2$

(c) $-x^4-x^3-2x^2-1+x+2x^3+x^2-6-4x+2+x^4$

The answers to the above exercise and the other exercises (in the boxes) to come can be found in pages 36 and 37 of this booklet.

Extra Exercises for Practice

Simplify the following expressions:

$(a)\ x^3 - xy - 4y^2 + 2xy - 3x^3 + \frac{1}{2}y^2$

$(b)\ 2x^2 + 3xy - 4y^2 - 2x^2 + y^2 - \frac{6}{2}xy$

$(c)\ -p^4 - p^3 - 2p^2 + p + 2p^3 + p^2 - 6 - 4p + 2 + p^4$

$(d)\ -3p + q - 7q + p - q - 5p + 2q$

$(e)\ (4q+1) - (q^3 - 3q^2 + 9q + 1)$

Other forms of Simplifications

Simplification can also be applied to fractions. For example when you divide 4 by 2 you get 2. This can be demonstrated using simplification:

$$\frac{4}{2} = \frac{2\times 2}{2} = \frac{2\times \cancel{2}}{\cancel{2}} = \frac{2\times 1}{1} = \frac{2}{1} = 2$$

Likewise when you divide 3 by 6 you get 1/2. We can also show this by using simplification:

$$\frac{3}{6} = \frac{3}{3\times 2} = \frac{\cancel{3}}{\cancel{3}\times 2} = \frac{1}{1\times 2} = \frac{1}{2}$$

Let us do another example by simplifying 18/24:

$$\frac{18}{24}=\frac{9\times\cancel{2}}{12\times\cancel{2}}=\frac{9}{12}$$

However, stopping there will not give you the answer. Because $\frac{9}{12}$ can further be simplified.

$$\frac{9}{12}=\frac{3\times 3}{4\times 3}=\frac{3\times\cancel{3}}{4\times\cancel{3}}=\frac{3}{4}$$

Exercise 1.1.2

Simplify the following expressions:

$a)\ \frac{24}{18}\quad b)\ \frac{11}{33}\quad c)\ \frac{27}{6}\quad d)\ \frac{16}{9}\quad e)\ \frac{24}{36}$

Extra Exercises for Practice

Simplify these expressions

$(a)\ \frac{9}{36};\ (b)\ \frac{32}{72};\ (c)\ \frac{90}{150};\ (d)\ \frac{112}{28};\ (e)\ \frac{60}{72}$

So far in the simplification with fractions we have only dealt with numbers. It is also possible to simplify fractions that contain letters. For example, let us simplify $\frac{a\times b}{b}$.

$$\frac{a\times b}{b}=\frac{a\times\cancel{b}}{\cancel{b}}=\frac{a\times 1}{1}=\frac{a}{1}=a\ .$$

Let us do another example by simplifying $\frac{a\times b}{a\times b}$. As you can see there is a slight difference between this example and the one above.

$$\frac{a\times b}{a\times b}=\frac{a\times\cancel{b}}{a\times\cancel{b}}=\frac{a\times 1}{a\times 1}=\frac{\cancel{a}}{\cancel{a}}=\frac{1}{1}=1$$

Of course this could have been done without going through all the steps:

$$\frac{\cancel{a}\times\cancel{b}}{\cancel{a}\times\cancel{b}}=1.$$

Let us try another example by combining this time letters and numbers. Let us try to simplify $\frac{12ab}{3b}$. We can start by simplifying the numbers first or the letters. Let us start with the letters.

$\frac{12a\cancel{b}}{3\cancel{b}}=\frac{12a}{3}$. Now the next step is to see whether the numbers can also be simplified.

$$\frac{12a}{3}=\frac{3\times 4\times a}{3}=\frac{\cancel{3}\times 4\times a}{\cancel{3}}=\frac{1\times 4\times a}{1}=\frac{4\times a}{1}=4a.$$

Again this could have been done in one go if you are familiar with fractions.

$$\frac{\overset{4}{\cancel{12}}\,a\cancel{b}}{\cancel{3}\cancel{b}}=4a$$

Using the same method as above let us try to simplify the following expressions (a) $\frac{a^2}{a}$; (b) $\frac{a^5}{a^3}$; (c) $\frac{x^5}{-x^2}$ and (d) $\frac{6xy}{2x^2}$.

(a) $\frac{a^2}{a}=\frac{\cancel{a}\times a}{\cancel{a}}=\frac{1\times a}{1}=a$

(b) $\frac{a^5}{a^3} = \frac{\overbrace{a\times a\times a\times a\times a}^{5times}}{\underbrace{a\times a\times a}_{3times}} = \frac{\cancel{a}\times\cancel{a}\times\cancel{a}\times a\times a}{\cancel{a}\times\cancel{a}\times\cancel{a}} = \frac{a\times a}{1} = a\times a = a^2$.

This expression could have also been solved in one go, as follows: $\frac{a^{\cancel{5}^{\,2}}}{\cancel{a}^3} = a^2$

(c) $\frac{x^5}{-x^2} = \frac{x\times x\times x\times x\times x}{-x\times x} = \frac{\cancel{x}\times\cancel{x}\times x\times x\times x}{-\cancel{x}\times\cancel{x}} = \frac{x\times x\times x}{-1} = x\times x\times x = -x^3$.

Or $\frac{x^5}{-x^2} = \frac{x^{\cancel{5}^{\,3}}}{-\cancel{x}^{\cancel{2}}} = -x^3$

(d) $\frac{6xy}{2x^2} = \frac{6xy}{2x\times x} = \frac{6\cancel{x}y}{2\cancel{x}\times x} = \frac{6y}{2\times x} = \frac{\cancel{2}\times 3\times y}{\cancel{2}\times x} = \frac{3\times y}{1\times x} = \frac{3y}{x}$.

Or $\frac{\overset{3}{\cancel{6}}\,\cancel{x}y}{\cancel{2}x^{\overset{1}{\cancel{2}}}} = \frac{3y}{x}$.

Hint:

When simplifying a fraction only common factors to both the numerator and denominator are cancelled.

Exercise 1.1.3

Simplify

$(a)\ \frac{9a}{3b};\ (b)\ \frac{18xy}{6y};\ (c)\ \frac{x^3}{x};(d)\ \frac{-x^5}{-x^2};\ (e)\ \frac{x^5}{x^2}$

Extra Exercises for Practice

Simplify

$(a)\ \frac{18xy}{4x}$; $(b)\ \frac{36q}{6q}$; $(c)\ \frac{p^3}{p^3}$; $(d)\ \frac{-y^5}{-y^2}$; $(e)\ \frac{y^4}{y^3}$

1.2 Expansion of Brackets

Expanding brackets means multiplying terms to remove the brackets. Brackets expansion might in some cases involve simplification, which we have just covered. As before, we will proceed through examples.

Let us try to expand the following algebraic expressions: $(a)\ 2(3x-2y)$; $(b)\ 2x(3x+2y)$ and $(c)-3(4x-1)$.

$(a)\ 2(3x-2y)=(2\times 3x)-(2\times 2y)=6x-4y$

$(b)\ 2x(3x+2y)=(2x\times 3x)+(2x\times 2y)=6x^2+4xy$

$(c)-3(4x-1)=(-3\times 4x)-(-3\times 1)=(-12x)-(-3)=-12x+3$

Hint:

To expand an expression, with simple brackets, we multiply each term inside the brackets by the term outside the brackets.

Now let us try to expand the following algebraic expressions: $(2x+5)(3x-4)$.

The first thing to do is to multiply each term of the first brackets by the second brackets, i.e.:

$(2x + 5)(3x-4) = 2x(3x-4)+5(3x-4)$.

From the double brackets our problem is reduced to expanding simple brackets. Following the same steps as above this leads us to:

$$2x(3x-4)+5(3x-4) = \underbrace{\{(2x\times 3x)-(2x\times -4)\}}_{2x(3x-4)} + \underbrace{\{(5\times 3x)-(5\times 4)\}}_{5(3x-4)}$$

The terms in the right hand side can further be expanded:

$$\{(2x\times 3x)-(2x\times -4)\} + \{(5\times 3x)-(5\times 4)\} = \{6x^2 - 8x\} + \{15x-20\} = 6x^2 - 8x + 15x - 20$$

Now it is time to use one of the tools you have learned so far i.e. simplification. As you can see two terms $8x$ and $15x$ are like terms (i.e. similar). Therefore, we need to put them together so that no term is repeated more than once in our final expression.

$\{6x^2 - 8x + 15x - 20\} = 6x^2 + 7x - 20$, which is our final expression.

Again those who are comfortable with brackets expansion do not have to go through the different steps. These steps have been followed to help you understand how we reach the final expression.

We can expand the same expression using a short cut, as bellow:

$$(2x + 5)(3x-4) = \underbrace{6x^2}_{2x\times 3x} \underbrace{-8x}_{2x\times(-4)} \underbrace{+15x}_{5\times 3x} \underbrace{-20}_{5\times(-4)} = 6x^2 + 7x - 20 .$$

Exercise 1.2.1

Expand these algebraic expressions:

$(a)\ 5(x+3)$; $(b)\ x(x-6)$; $(c)\ 3a(5x-2)$; $(d)\ (2y-3)(5y+7)$;
$(e)\ (0-9x)(5x-4)$; $(f)\ (a+b)(c+d)$; $(g)\ (x-5)(x-2)$.

Extra Exercises for Practice

Expand

$(a)\ (4q+q^3+1)-(q^3-3q^2+9q+1)$

$(b)\ x^2(x-1)$

$(c)\ (z+1)(z-1)$

$(d)\ (a-b)(a-b)$

$(e)\ \dfrac{x}{3}(1-\dfrac{3}{x})$

1.3 Factorisation

You have just learned how to expand algebraic expressions with brackets. Factorisation works the other way round i.e. you are given an expanded expression and you have to, generally, rewrite it in a shorter form using brackets. We have two types of factorisation: simple and quadratic. However, factorisation of quadratic forms will be dealt with in the section reserved to solving quadratic equations. In other words, we will only be factorising simple algebraic expressions in this section. Factorisation is quite useful in solving some equations.

Let us start from the first example used in the "expansion" section. Expanding the expression: $2(3x-2y)$ led to the following expression: $6x-4y$. Now, factorisation will require that we transform the expression $6x-4y$ into $2(3x-2y)$. This could be done by following these steps.

1. We need to recognise the highest common factor in the expression: $6x-4y$. The highest common factor between the two terms $6x$ and $-4y$ is 2.
2. We need to put this highest common factor outside the brackets: $6x-4y=2(\bullet)$, where the dot ($\bullet$) represents the expression we have to put in the brackets so that when we multiply what is outside the brackets by what is inside them we get our previous expression.
3. We need to find the number which when multiplied by the highest common factor (2 in the present case) will give $6x$: $6x-4y=2(3x\bullet)$.
4. We need to find a second number which when multiplied by the highest common factor will give $-4y$: $6x-4y=2(3x-2y)$.

$$\exp ansion \rightarrow 2(3x-2y)=6x-4y$$
$$factorisation \rightarrow 6x-4y=2(3x-2y)$$

Let us try another example by factorising (a) $21x-28$; (b) $33x+11$; and (c) $\frac{1}{4}x-\frac{1}{2}$.

Hint:

To factorise an algebraic expression we must pull out the highest common factor between the terms and put it outside the brackets.

(a) $21x-28=\{\underbrace{\boxed{7}\times 3x}_{21x}\}+\{\underbrace{\boxed{7}\times -4}_{-28}\}=\boxed{7}(3x-4)$, where the frames ($\boxed{}$) highlight the highest common factor between the terms.

(b) $33x+11=\{\underbrace{\boxed{11}\times 3x}_{33x}\}+\{\underbrace{\boxed{11}\times 1}_{+11}\}=\boxed{11}(3x+1)$

(c) $\frac{1}{4}x-\frac{1}{2}=\left\{\underbrace{\boxed{\frac{1}{2}}\times\frac{1}{2}x}_{\frac{1}{4}x}\right\}+\left\{\underbrace{\boxed{\frac{1}{2}}\times -1}_{-\frac{1}{2}}\right\}=\boxed{\frac{1}{2}}\left(\frac{1}{2}x-1\right)$

So far, the highest common term in the expressions we have factorised are numbers. However this is not always the case. We might get expressions where the highest common number is a term. Again, let us go back to the second expression we expanded earlier. Remember: $2x(3x+2y)=6x^2+4xy$ after expansion. Now let us try to factorise $6x^2+4xy$.

$$6x^2+4xy=\{\underbrace{6x\times x}_{6x^2}\}+\{\underbrace{4x\times y}_{4xy}\}=\{\underbrace{3\times\boxed{2}\times\boxed{x}\times x}_{6x\times x}\}+\{\underbrace{\boxed{2}\times 2\times\boxed{x}\times y}_{4x\times y}\}.$$

$$=\boxed{2}\times\boxed{x}(3\times x+2\times y)=\boxed{2}\boxed{x}(3x+2y)$$

You don't have to go through these steps to get your answer. This is just to show you how we arrived to our solution. This question could have been answered in one go, as bellow.

$6x^2 + 4xy = 2x(3x + 2y)$.

Now let us try to factorise (a) $x^2y - xy^2$ and (b) $x^3 + 7x^2$.

(a) $x^2y - xy^2 = \underbrace{\{\boxed{x} \times x \times \boxed{y}\}}_{x^2y} + \underbrace{\{\boxed{x} \times \boxed{-y} \times y\}}_{-xy^2} = \boxed{x}\boxed{y}(x - y) = xy(x - y)$.

Or in one go: $x^2y - xy^2 = xy(x - y)$.

(b) $x^3 + 7x^2 = \underbrace{\{\boxed{x} \times \boxed{x} \times x\}}_{x^3} + \underbrace{\{7 \times \boxed{x} \times \boxed{x}\}}_{7x^2} = \boxed{x}\boxed{x}(x + 7) = x^2(x + 7)$

Or in one go $x^3 + 7x^2 = x^2(x + 7)$

Exercise 1.3.1

Factorise

$(a)\ 5x - 25$; $(b)\ 27x + 18$; $(c)\ -8x + 24$; $(d)\ x^2 + 4x$; $(e)\ x^2y + xy^2$;

$(d)\ x^2 + 4x$; $(e)\ x^2y + xy^2$; $(f)\ 4a^2x - 4a$; $(g)\ 24a^2x + 8a$

Extra Exercises for Practice

Factorise

$(a)\ a^2 - 2ab + b^2$

$(b)\ x^3 - x^2$

$(c)\ z^2 - 1$

$(d)\ p^2 + p - 2$

$(e)\ 2q^2 - \frac{18}{9}q^2$

2. Solving Equations

We will try to solve three types of equations: linear, quadratic, and simultaneous. For this section we might need to use some of the tools we have learned in the previous sections.

2.1 Solving Linear Equations

A linear equation is one that can be written in the form $ax+b=0$ where a and b are numbers and the unknown quantity is x. For example $4x+1=0$ and $\frac{1}{2}x-3=0$ are both linear equations. Linear equations may also appear in the following forms (which appeared to be different from $ax+b=0$ but are equivalent): $2x-1=3$; $-x+3=4x+1$; and $3(-2x+6)=0$. These equations can be rearranged to get the form $ax+b=0$, as shown bellow.

$2x-1=3$ → *Subtract 3 from both sides so that the right hand side becomes 0*

$2x-1-3=3-3$ → *Simplify this expression*

$2x-4=0$ → *This expression is of the form* $ax+b=0$

Now let us move to the second expression: $-x+3=4x+1$

$-x+3=4x+1$ → *Simplify this expression by bearing in mind that when you move on the other side of the "=" you change sign change*

$-x-4x=1-3 \Rightarrow -5x=-2$ → *Add +2 to both sides so that the right hand side is 0*

$-5x+2=-2+2$ → *Simplify*

$-5x+2=0$ → *This expression is of the form* $ax+b=0$

Finally, $3(-2x+6)=0$ can also be expressed in the form $ax+b=0$.

$3(-2x+6)=0$ → *Expand the brackets*

$-6x+18=0$ → *This expression is of the form* $ax+b=0$

Now let us try to solve some linear equations: (a) $x+10=0$; (b) $-5x+100=0$; (c) $4x-6=-3$; (d) $3(x+2)=5$; and (e) $5x+2=-2x-3$.

(a) $x+10=0 \Rightarrow x=-10$ (as you can see if you replace x by –10 you get 0)

(b) $-5x+100=0 \Rightarrow -5x=-100 \Rightarrow x=\dfrac{-100}{-5}=20$

(c) $4x-6=-3 \Rightarrow 4x=-3+6 \Rightarrow 4x=3 \Rightarrow x=\dfrac{3}{4}$

(d) $3(x+2)=5 \Rightarrow 3x+6=5 \Rightarrow 3x=5-6 \Rightarrow 3x=-1 \Rightarrow x=\dfrac{-1}{3}$

(e) $5x+2=-2x-3 \Rightarrow 5x+2x=-3-2 \Rightarrow 7x=-5 \Rightarrow x=\dfrac{-5}{7}$.

Here are some examples for you to work out.

Exercise2.1.1

Solve the following linear equations:

$(a)\ 3x-9=0;\ (b)\ -2x=24;\ (c)\ -10x=0;(d)\ 3x-13=9;$

$(e)\ 4x-20=3x+16;\ (f)\ 10(x+4)=0;\ (g)\ 6(x-3)=9;$

$(h)\ 2(4-3x)=-5x;\ (i)\ 4(-2x+1)=3(-5x-4);$

$(j)\ -\frac{1}{2}x=0;\ (k)\ \frac{2}{3}x=\frac{-1}{5};\ (l)\ 2x^2-3x-2=0.$

Extra Exercises for Practice

Solve these equations

$(a)\ 18p-9=0;\ (b)\ -4x-24=0;\ (c)\ -4y=0;(d)\ 3q-2=9;\ (e)\ \frac{2}{5}x=\frac{-1}{5}$

2.2 Solving Quadratic Equations

A quadratic equation is an equation of the form $ax^2+bx+c=0$ where $a,\ b\ and\ c$ are numbers, with $a\neq 0$. We have two sorts of quadratic equations. In the first case $a=1$ i.e. the quadratic equation takes the form: $x^2+bx+c=0$. In the second case $a\neq 1$. We will start with the first case and then after we will go on to look at the second case.

2.2.1 Quadratic equations when $a=1$

Before this type of equation can be solved we need to factorise it (remember factorising means to put in brackets), a tool you have learned earlier. Starting by a simple example would perhaps make things easier. Let us say we were asked to solve the following quadratic equation:

$x^2+3x-10=0$ where $a=1,\ b=3,\ and\ c=\text{-}10$.

Remember we said earlier that before trying to solve this form of equation we must try to factorise it first, i.e. we have to rewrite the expression in brackets form, as below:

$x^2+3x-10=0 \Rightarrow (x\ \bullet)(x\ \circ)=0$.

As you could notice, in addition to the xs there are 2 circles (one dark the other clear) inside the brackets. The idea is to find 2 numbers (represented by the 2 circles) such that:

1. when you multiply the 2 numbers represented by the circles you get -10
2. when you add them up you should get $+3$

There are few possibilities:

$(a)\ -10\times 1=-10\ but\ -10+1\ is\ not\ equal\ to\ +3$
$(b)\ -5\times 2=-10\ but\ -5+2\ is\ not\ equal\ to\ +3$
$\boxed{(c)\ -2\times 5=-10\ and\ -2+5\ is\ equal\ to\ +3}$
$(d)\ -1\times 10=-10\ but\ -1+10\ is\ not\ equal\ to\ +3$

As you could see it is only option (c) that satisfies our condition. In other words, we should replace the dark circle by –2 and the clear circle by +5, which gives us:

$x^2+3x-10=0 \Rightarrow (x-2)(x+5)=0$ (for practice try to expand this expression)

Now that we have factorised our expression we can easily solve it.

$$(x-2)(x+5)=0 \Rightarrow \begin{cases} (x-2)=0 \Rightarrow \underbrace{x-2=0}_{linearequation} \Rightarrow x=2 \\ \qquad\qquad\qquad or \\ (x+5)=0 \Rightarrow \underbrace{x+5=0}_{linearequation} \Rightarrow x=-5 \end{cases}$$

The solutions to our problem are either $x=2$ or $x=-5$. As you could see we have transformed our quadratic equation to 2 linear equations.

Now let us check and see if our solution is right. This is done by replacing x in the expression $x^2+3x-10=0$ by 2 or –5.

If $x=2$ then we have:

$$\begin{array}{l} x^2+3x-10=0 \\ \Downarrow \quad \Downarrow \quad \Downarrow \\ 2^2+3\times 2-10=4+6-10=10-10=0 \end{array}$$

If $x=-5$ then we have:

$x^2 + \quad 3x \qquad -10 = 0$

$\Downarrow \qquad \Downarrow \qquad\quad \Downarrow$

$(-5)^2 + 3\times(-5) \; -10 \; = 25 \; -15 \; -10 \; = 25 - 25 = 0$

So, in both cases we get 0, which means we can be satisfied with our results.

However, there is something that is worth pointing out. A lot of students fail to remember that when you square a negative number it becomes positive. For example, $(-5)^2 = 25$ and **<u>not</u>** $(-5)^2 = -25$. If you are trying to calculate this using a calculator always put the negative number inside the brackets, as done here, before raising it to the power 2. This is because in mathematics the power operation comes before the multiplication, division, addition and subtraction. For example if you write -5^2 the calculator will first calculate 5^2, which is 25 and then it would multiply by –1, which is –25. Some calculators will understand that -5^2 is the same as $(-5)^2$; but to be on the safe side always use the brackets.

Now coming back to our quadratic equations let us try to do another example by solving the following equation: $x^2 - x - 56 = 0$.

As before we must factorise the expression. We need to find 2 numbers such that when multiplied the result is –56. There are 4 possibilities: –56 and 1; -8 and 7; -7 and 8; and –1 and 56. However, among these 4 possibilities only -8 and 7 add up to –1. Therefore:

$$x^2 - x - 56 = 0 \Rightarrow (x-8)(x+7) = 0$$

$$(x-8)(x+7)=0 \Rightarrow \begin{cases} (x-8)=0 \Rightarrow \underbrace{x-8=0}_{linear equation} \Rightarrow x=8 \\ \\ \qquad\qquad or \\ \\ (x+7)=0 \Rightarrow \underbrace{x+7=0}_{linear\ \ equation} \Rightarrow x=-7 \end{cases}$$

We have 2 solutions: $x=8$ or $x=-7$. You can check to see if the answer is correct, as we did previously.

Finally let us try this example by solving: $x^2-3x=0$. This of course a quadratic equation, with $c=0$. Remember we have said that to be a quadratic equation only a must not be equal to 0. So if we apply the same concept as above we should find 2 numbers such that when multiplied the result is 0 (because $c=0$). There are 2 possibilities: -3 and 0; 0 and 3. However, the only case that adds up to –3 is the first one, i.e. –3 and 0. So therefore we can factorise our expression as follows:

$$x^2-3x=0 \Rightarrow (x-3)(x+0)=0 \Rightarrow (x-3)(x)=0 \Rightarrow x(x-3)=0$$

Now that we have factorised our expression we can solve it.

$$x(x-3)=0 \Rightarrow \begin{cases} x=0 \Rightarrow \underbrace{x=0}_{linear equation} \Rightarrow x=0 \\ \\ \qquad\qquad or \\ \\ (x-3)=0 \Rightarrow \underbrace{x-3=0}_{linear equation} \Rightarrow x=3 \end{cases}$$

We, therefore, have 2 solutions: $x=0$ or $x=3$.

Exercise 2.2.1
Solve the following quadratic equations

$(a)\ x^2 - 6x - 16 = 0$; $(b)\ x^2 + 9x + 20 = 0$; $(c)\ x^2 - 8x + 15 = 0$;
$(d)\ x^2 + 3x - 40 = 0$; $(e)\ x^2 - 12x = 0$

Extra Exercises for Practice

Solve these equations

$(a)\ x^2 - 5x - 14 = 0;\ (b)\ x^2 - 6x + 9 = 0;\ (c)\ y^2 + 6y - 10 = 0;$
$(d)\ a^2 - 16 = 0;\ (e)\ p^2 - 6p = 0$

2.2.2 Quadratic equations with $a \neq 1$

In this form of quadratic equations a is a number, which is greater than one in absolute term. Although in some cases these equations can be solved using the same technique as above in others it might not be easy to do so. If $ax^2 + bx + c = 0$ with $a \neq 1$ then we can calculate what is known as the quadratic formula. The solution for that equation is:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad or\ if\ \ you\ prefer \begin{cases} x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a} \\ \qquad\qquad or \\ x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a} \end{cases}$$

Now let us try the following example: $2x^2 + 3x - 5 = 0$.

$$2x^2 + 3x - 5 = 0$$

$$\begin{array}{lll} \downarrow & \downarrow & \downarrow \\ a & b & c \end{array}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{3^2 - 4 \times 2 \times (-5)}}{2 \times 2} = \frac{-3 \pm \sqrt{9+40}}{4} = \frac{-3 \pm \sqrt{49}}{4} = \frac{-3 \pm 7}{4}$$

$$\Rightarrow \begin{cases} x = \frac{-3+7}{4} = \frac{4}{4} = 1 \\ or \\ x = \frac{-3-7}{4} = \frac{-10}{4} = \frac{-5}{2} \end{cases}$$

We have therefore two solutions: $x = 1 \; or \; x = \frac{-5}{2}$.

Let us try another example: $4x^2 + 6x + 2 = 0$

$$4x^2 + 6x + 2 = 0$$

$$\begin{array}{lll} \downarrow & \downarrow & \downarrow \\ a & b & c \end{array}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-6 \pm \sqrt{6^2 - 4 \times 4 \times 2}}{2 \times 4} = \frac{-6 \pm \sqrt{36-32}}{8} = \frac{-6 \pm \sqrt{4}}{8} = \frac{-6 \pm 2}{8}$$

$$\Rightarrow \begin{cases} x = \frac{-6+2}{8} = \frac{-4}{8} = \frac{-1}{2} \\ or \\ x = \frac{-6-2}{8} = \frac{-8}{8} = -1 \end{cases}$$

We have also two solutions $x = \frac{-1}{2} \; or \; x = -1$.

Note: a quadratic equation in the form of $x^2 + bx + c = 0$ (where a = 1), which we have dealt with earlier, can also be solved by using the quadratic formula.

To show this let us use the first example that we solved earlier i.e.: $x^2 + 3x - 10 = 0$

$$\begin{array}{lll} x^2 + & 3x & -10 = 0 \\ \downarrow & \downarrow & \downarrow \\ a & b & c \end{array}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{3^2 - 4 \times 1 \times (-10)}}{2 \times 1} = \frac{-3 \pm \sqrt{9 + 40}}{2} = \frac{-3 \pm \sqrt{49}}{2} = \frac{-3 \pm 7}{2}$$

$$\Rightarrow \begin{cases} x = \frac{-3+7}{2} = \frac{4}{2} = 2 \\ or \\ x = \frac{-3-7}{2} = \frac{-10}{2} = -5 \end{cases}$$

We have 2 solutions: $x = 2$ *or* $x = -5$. If you check the solution we got earlier using the other method you will see that the results are exactly the same.

Exercise 2.2.2

Solve the following quadratic equations:

(a) $4x^2 + 5x + 1 = 0$; (b) $2y^2 - 2y - 4 = 0$; (c) $y^2 + 12y + 35 = 0$;

(d) $9a^2 - 12a - 5 = 0$

Extra Exercises for Practice

Solve these equations

(a) $4x^2 - 10x + 6 = 0$; (b) $3y^2 - 6y = 0$; (c) $12p^2 + 16p - 16 = 0$;
(d) $3x^2 - 6x = 0$; (e) $6a^2 + 15a = 18$

2.3 Solving Simultaneous Equations

Simultaneous equations are a system formed by two or more equations. However, we are only going to deal with systems of two equations here. Simultaneous equations can be used to solve problems in real life. For example:

- Maria goes to the supermarket and buys 4 CDs and 2 pairs of shoes for the total price of £80.
- Louise goes to the same shop and 3 CDs and 1 pair of shoes for the total price of £50.[1]

What is the price of a CD and what is that of a pair of shoes? By the time we finish studying this section you should be able to solve this problem. Some of you may not have to wait to do so.

There are three methods to solve simultaneous equations:

1. the substitution method
2. the elimination method
3. the graphical method

We will focus on the first two methods in this sub-section. Concerning the graphical method we will deal with it when we come to section 3 (reserved to graphs and function).

2.3.1 The substitution method

Using an example will make it easier to understand this method. Let us say we are trying to solve the following system of equations:

[1] We assume that Maria and Louise are buying exactly the same products.

$$\begin{cases} 4x - y = 0 \\ 3x + y = 7 \end{cases}$$

The first thing to do is to label the two equations with the first being labelled with (1) and the second with (2), as below:

$$\begin{cases} 4x - y = 0 & (1) \\ 3x + y = 7 & (2) \end{cases}$$

The second thing to do is to try to rearrange one of the equations such that one variable is expressed in terms of the other. For example, we could rearrange equation (1) in the following manner:[2]

$$4x - y = 0 \Rightarrow 4x = y \qquad (3)$$

The third thing to do is to substitute (3) into (2) i.e. we replace y by 4x in (2), which gives:

$$3x + 4x = 7 \Rightarrow 7x = 7 \Rightarrow x = \frac{\cancel{7}}{\cancel{7}} = 1 \qquad (4)$$

Finally, we put (4) into (3), which gives:

$$4 \times 1 = y \Rightarrow y = 4$$

Therefore the solution to our system of equations is $x = 1 \; and \; y = 4$. If we replace $x \; and \; y$ by their respective values we will see that the equations-(1) and (2)-hold. You could try to do this as an exercise.

Another example: let us solve the following system of equations:

$$\begin{cases} 2x + 4y = 6 \\ -4x - 3y = 3 \end{cases}$$

As before we need to label them:

[2] Try to solve the system but this time rearranging the second equation. You should arrive to the same results.

$$\begin{cases} 2x+4y=6 & (1) \\ -4x-3y=3 & (2) \end{cases}$$

In the example we did earlier, we rearranged equation (1). But you could also start by rearranging the second equation, (2). What you have to bear in mind is to rearrange the equation that is easier to rearrange. In the present case by looking at the two equations it seems that equation (1) might be easier to rearrange.

$$2x+4y=6 \Rightarrow 4y=6-2x \Rightarrow y=\frac{6-2x}{4}=\frac{\cancelto{3}{6}}{\cancelto{2}{4}}-\frac{\cancelto{1}{2}x}{\cancelto{2}{4}} \Rightarrow y=\frac{3}{2}-\frac{1}{2}x \quad (3)$$

Now let us put (3) into (2): what we have to do here is to replace $y \; by \; \frac{3}{2}-\frac{1}{2}x$:

$$-4x-3y=3 \Rightarrow -4x-3\underbrace{\left(\frac{3}{2}-\frac{1}{2}x\right)}_{y}=3 \Rightarrow -4x-\frac{9}{2}+\frac{3}{2}x=3 \Rightarrow -4x+\frac{3}{2}x=3+\frac{9}{2}$$

$$\Rightarrow \frac{2\times(-4x)+3x}{2}=\frac{2\times 3+9}{2} \Rightarrow \frac{-8x+3x}{2}=\frac{6+9}{2}$$

$$\Rightarrow \frac{-5x}{2}=\frac{15}{2} \Rightarrow -5x=15 \Rightarrow x=\frac{15}{-5}=-3 \qquad (4)$$

Now putting (4) in (3) we get:

$$y=\frac{3}{2}-\frac{1}{2}\times(-3) \Rightarrow y=\frac{3}{2}+\frac{3}{2} \Rightarrow y=\frac{6}{2}=3$$

Therefore, the solution to our system of equations is $x=-3 \; and \; y=3$.

Note: Now if you hate fractions you can write $\frac{3}{2}x \; as \; 1.5x \; and \; \frac{9}{2} as \; 4.5$. So you will have:

$$-4x+1.5x=3+4.5 \Rightarrow -2.5x=7.5 \Rightarrow x=\frac{7.5}{-2.5}=-3.$$

According to (3), $y=\frac{3}{2}-\frac{1}{2}x \Rightarrow y=1.5-0.5x$. $we \; know that \; x=-3$

$therefore \; y=1.5-0.5(-3)=1.5+1.5=3$.

2.3.2 The elimination method

In this method one has to subtract (or add) the equations from (or to) one another. Again let us try to solve our first example using the elimination method.

$$\begin{cases} 4x - y = 0 \\ 3x + y = 7 \end{cases}$$

As before we have to label these equations:

$$\begin{cases} 4x - y = 0 & (1) \\ 3x + y = 7 & (2) \end{cases}$$

Now we need to match up the numbers in front of the $x's \text{ or } y's$ so that when we subtract (or add) one equation from (or to) the other, one of the variable disappear. Again we need to choose the easiest option. Looking at the system we can see that by adding (1) to (2) the $y's$ disappear (or they are eliminated)

$$\begin{array}{l} 4x - y = 0 \quad (1) \\ \quad + \\ 3x + y = 7 \quad (2) \\ \hline 7x + 0 \ = 7 \end{array} \Rightarrow 7x = 7 \Rightarrow x = \frac{7}{7} = 1$$

We need now to replace x by its value in any of the equations (it does not matter which one; but always choose the easiest one to calculate). Looking at the system the easiest one to calculate is the first equation, (1).

$4x - y = 0$ *and* we found that $x = 1 \Rightarrow 4 \times 1 - y = 0 \Rightarrow 4 - y = 0 \Rightarrow 4 = y$

As you can see we arrived to the same results as using the substitution method.

We can also try our second example using the elimination approach.

$$\begin{cases} 2x + 4y = 6 \\ -4x - 3y = 3 \end{cases}$$

Again we have to label these equations:

$$\begin{cases} 2x+4y=6 & (1) \\ -4x-3y=3 & (2) \end{cases}$$

Now we need to match the numbers either in front of the $x's$ or $y's$ so that we can eliminate one variable. Looking at the system it is perhaps easier to try to multiply (1) by 2, so that when we add up this new equation (that we will call (3)) to (2) we should be able to eliminate the $x's$.

$$\begin{cases} 2x+4y=6 & (1)\ \times 2 \\ -4x-3y=3 & (2) \end{cases} \Rightarrow \begin{cases} 4x+8y=12 & (3) \\ \quad + & \\ -4x-3y=3 & (2) \end{cases}$$

$$0 \quad +5y=15 \Rightarrow y=\frac{15}{5}=3$$

To find the value of x, we have to replace y by its value, 3, in one of the original equations of the system. As remarked earlier, it does not matter which equation you choose. However, choosing the easiest one to calculate will minimise the risk of making mistakes. From (1) and (2) there is not much to choose from. We, however, decide to choose (1) to the find the value of x; this gives us:

$$2x+4y=6 \Rightarrow 2x+4\times 3=6 \Rightarrow 2x+12=6 \Rightarrow 2x=6-12 \Rightarrow 2x=-6 \Rightarrow x=\frac{-6}{2}=-3$$

As before, the solution to our system of equation is $x=-3$ *and* $y=3$.

Exercise 2.2.3

Solve the following system of equations using both the substitution and elimination methods:

(a) $2x-y=-3$ *and* $-4x-y=-6$; (b) $-2x+y=-3$ *and* $5x-2y=8$;

(c) $5x+4y=13$ *and* $3x+8y=5$; (d) $4x=12-8y$ *and* $-6-8x=6y$.

Extra Exercises for Practice

Solve these equations using the substitution and elimination methods

$(a)\ \begin{cases} 4p+3q=24 \\ 2p+3q=18 \end{cases};\ (b)\ \begin{cases} 4x+2y=16 \\ 2x+2y=20 \end{cases};\ (c)\ \begin{cases} 4q+3p=24 \\ 2p-q=12 \end{cases};$

;

$(d)\begin{cases} \frac{1}{2}x+\frac{3}{2}y=6 \\ x+y=3 \end{cases};$

(e) Marie buys 3 cans of coke and 6 chocolate bars for £5.50. Alan goes to the same shop and buys 6 cans of coke and 12 chocolate bars for £11.00. Assuming that both buy the same item, derive the price for a can a coke and the price for a chocolate bar using the substitution or elimination method.

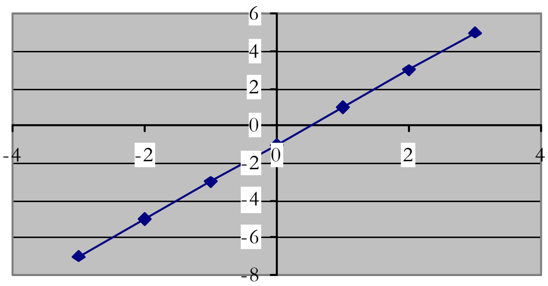
3. Graphs

In this section, we are only going to deal with graphs of linear function. A function f is said to be linear when $f(x)=ax+b$, where a and b are numbers. a is called the gradient and b is called the intercept. This linear function is sometimes written as: $y=ax+b$. As you might have realised there is a slight resemblance between a linear equation and a linear function. Remember a linear equation is of the form: $ax+b=0$. In other words, if $f(x)$ or y is set equal to zero a linear function becomes a linear equation.

3.1 Plotting Linear Graphs

When plotting linear graphs it is always a good thing to draw a table. Let us try to plot the following function: $f(x)=2x-1$. To do this, you only have to choose the values for x and then using your calculator you calculate the value of $f(x)$, for each x that you have chosen. You are free to choose any value of x you like. However, as in real life, always try to simplify things by remembering that if you choose big values of x your are likely to find yourself doing complicated calculation and more importantly it might be difficult for you to plot your graph on a A4 paper sheet. It is also important to bear in mind that when choosing the values of x you need to choose both negative and positive integers.

x	-3	-2	-1	0	1	2	3
$f(x)$	-7	-5	-3	-1	1	3	5



Although we have chosen 7 values for x it is possible to draw the same graph using only two values for x and finding the corresponding values for $f(x)$. This is because this function is linear. Put differently, when drawing a graph for a linear function we only have to find 2 points on the map and then join them.[3] However, to be sure that you have not done any mistakes during your calculations it might be better to consider 3 points so that if your line

[3] This only applies to linear functions.

passes through these 3 points you can be reassured that you have done the right thing.

3.2 Graphs and system of simultaneous equations

We have already solved systems of simultaneous equations using the substitution and the elimination methods. They can also be solved using the graphical method. The best way to proceed is to repeat the same examples that we used for the substitution and elimination methods.

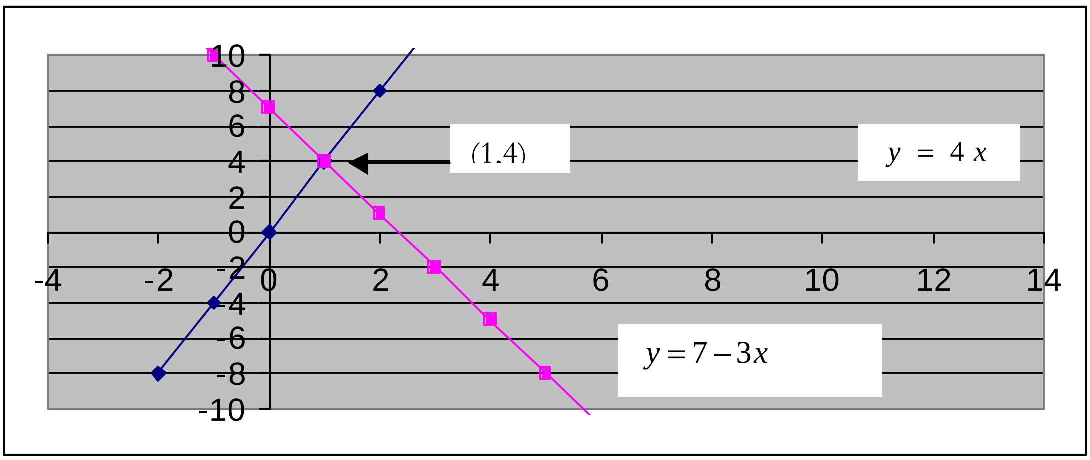
The first example we used was:

$$\begin{cases} 4x - y = 0 \\ 3x + y = 7 \end{cases}$$

In order to be able to solve this system we have to rewrite the system in a way that y is expressed in terms of x in both equations, as follows:

$$\begin{cases} 4x - y = 0 \Rightarrow \\ 3x + y = 7 \Rightarrow \end{cases} \begin{cases} y = 4x \\ y = 7 - 3x \end{cases}$$

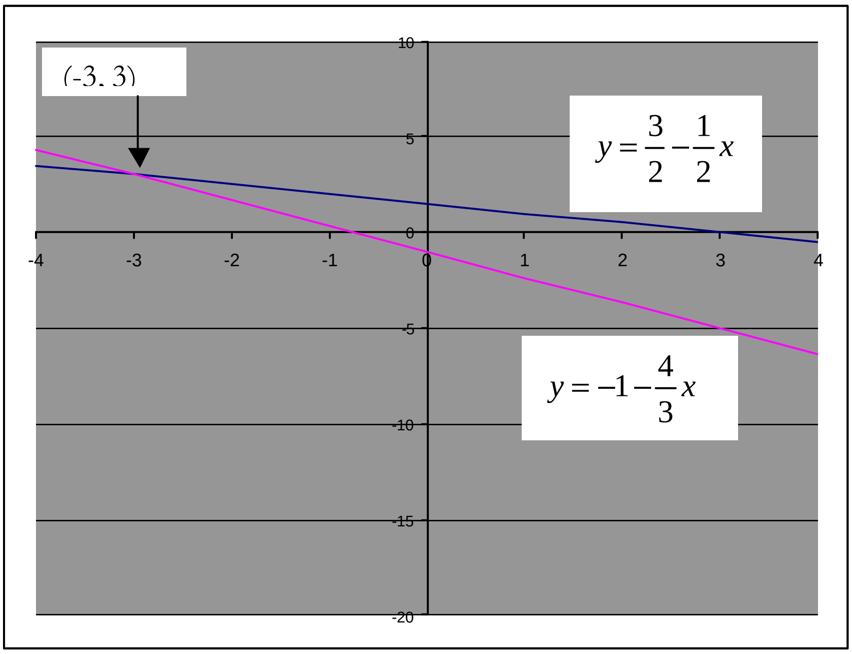
Now what we have to do is to plot these two graphs and their intersection should give us the solution to our problem.



According to the plot of the two graphs the solution to our problem is $x = 1 \; and \; y = 4$, which is exactly what we got earlier. However, we must admit we have cheated a little bit by plotting the graphs using a mathematical software. However, imagine we have to plot these graphs by hand it will not be easy to get the exact solution, even with graphical papers sometimes. This is one drawback of using the graphical method to solve simultaneous equations.

We can also try to solve our second problem by graphical method.

$$\begin{cases} 2x + 4y = 6 \Rightarrow \\ -4x - 3y = 3 \Rightarrow \end{cases} \begin{cases} 4y = 6 - 2x \Rightarrow \\ -3y = 3 + 4x \Rightarrow \end{cases} \begin{cases} y = \dfrac{6}{4} - \dfrac{2x}{4} \Rightarrow \\ y = \dfrac{3}{-3} + \dfrac{4x}{-3} \Rightarrow \end{cases} \begin{cases} y = \dfrac{3}{2} - \dfrac{1}{2}x \\ y = -1 - \dfrac{4}{3}x \end{cases}$$



From the plot it appears that the intersection of the graphs is at $x = -3 \; and \; y = 3$, as before.

Exercise 3.1

Using the graphical method solve the system of simultaneous equations in exercise 2.2.3.

Extra Exercises for Practice

Solve these equations using the graphical method and compare your results with your previous answers

$$(a)\ \begin{cases} 4p+3q=24 \\ 2p+3q=18 \end{cases};\ (b)\ \begin{cases} 4x+2y=16 \\ 2x+2y=20 \end{cases};\ (c)\ \begin{cases} 4q+3p=24 \\ 2p-q=12 \end{cases};$$

$$(d) \begin{cases} \frac{1}{2}x+\frac{3}{2}y=6 \\ x+y=3 \end{cases};$$

(*e*) *Marie buys* 3 *cans of coke and* 6 *chocolate bars for* £5.50. *Alan goes to the same shop and buys* 6 *cans of coke and* 12 *chocolate bars for* £11.00. *Assuming that both buy the same items, derive the price for a can of coke and the price for a chocolate bar using the substitution or elimination method.*

Topic 2 NUMBERS

1. Fractions

We have come across fractions in almost every section of the previous topic. This tells us that understanding fractions is important in solving many mathematical problems. A fraction is a mathematical expression of the form: $\frac{a}{b}$, where $a \text{ is the numerator}$ and $b \text{ is the denomiator}$. Although a can be any number (including zero) b cannot be zero because dividing by zero is not possible.[4]

1.1 Equivalent Fractions

Suppose that we are given the following fraction: $\frac{1}{2}$. Multiplying the numerator and denominator by the same number does not change the value of the fraction, as shown below.

$\frac{1}{2}=\frac{1\times 2}{2\times 2}=\frac{2}{4}$ or $\frac{1}{2}=\frac{1\times 100}{2\times 100}=\frac{100}{200}$

$\frac{1}{5}=\frac{1\times 3}{5\times 3}=\frac{3}{15}$ or $\frac{1}{5}=\frac{1\times 1000}{5\times 1000}=\frac{1000}{5000}$

Similarly, dividing a fraction by the same number does not change the value of the fraction.

$$\frac{4}{6}=\frac{4\div 2}{6\div 2}=\frac{2}{3}$$

$$\frac{18}{72}=\frac{18\div 9}{72\div 9}=\frac{2}{8}=\frac{2\div 2}{8\div 2}=\frac{1}{4}$$

[4] Using your calculator try to divide any number by zero: you should get *Error* on the screen

Not*e:* $\frac{A}{B}=\frac{C}{D}$ *does not mean* $A=C$ *and* $B=D$. *For example,* $\frac{1}{5}=\frac{1000}{5000}$ *but 1 is not equal 1000 and 5 is not equal to 5000.*

1.2 Adding and Subtracting Fractions

Only fractions with the same denominator can be added and subtracted. In other words, if we are given two fractions, which do not have the same denominator, we need to rewrite the fractions so that they all have the same denominator.[5] For example:

$$\frac{3}{5}+\frac{4}{7}=\frac{3\times 7}{5\times 7}+\frac{4\times 5}{7\times 5}=\frac{21}{35}+\frac{20}{35}$$

Both fractions have now the same denominator and, therefore, we can add them up:

$$\frac{21}{35}+\frac{20}{35}=\frac{21+20}{35}=\frac{31}{35}$$

We only add up the numerators (and not the denominators).

Let us try another example this time with a subtraction:

$$\frac{5}{6}-\frac{1}{18}=\frac{5\times 18}{6\times 18}-\frac{1\times 6}{18\times 6}=\frac{90}{108}-\frac{6}{108}$$

[5] You might have come across the word lowest common denominator (LCD).

We can do the subtraction now that both fractions have the same denominator

$$\frac{90}{108}-\frac{6}{108}=\frac{90-6}{108}=\frac{84}{\underbrace{108}_{divide\,both\,by\ 2}}=\frac{42}{\underbrace{54}_{divide\,both\,by\ 2}}=\frac{21}{\underbrace{27}_{divide\,both\,by\ 3}}=\frac{7}{9}$$

Exercise 1.1

Evaluate the following fractions and simplify where appropriate

$(a)\ \frac{2}{3}-\frac{5}{9};\ (b)\ \frac{4}{6}+\frac{2}{8};\ (c)\ \frac{20}{12}+\frac{12}{20};\ (d)\ \frac{1}{9}+\frac{11}{12};\ (e)\ \frac{2}{3}-\frac{5}{9}+\frac{11}{12}$

Extra Exercises for Practice

Evaluate

$(a)\ \frac{4}{3}-\frac{10}{9};\ (b)\ \frac{x}{6}+\frac{2}{5};\ (c)\ \frac{20p}{12}+\frac{12}{20p};\ (d)\ \frac{1}{y}+\frac{11}{12};\ (e)\ \frac{3}{5}-\frac{5}{9}+\frac{48}{12}$

1.2 Multiplying Fractions

To multiply two or more fractions:

1. Multiply the numerators to form a new numerator
2. Multiply their denominator to form a new numerator

Let us try some examples:

$\frac{3}{5}\times\frac{4}{7}=\frac{3\times4}{5\times7}=\frac{12}{35};$

$$\frac{5}{8}\times\frac{4}{5}=\frac{5\times4}{8\times5}=\frac{20}{40}=\frac{1}{2};$$

$$\frac{5}{8}\times7=\frac{5}{8}\times\frac{7}{1}=\frac{5\times7}{8\times1}=\frac{35}{8}\ \textit{or if you prefer}\ 4\,{}^{3}\!/_{8};$$

Exercise 1.2

Evaluate the following:

$(a)\ \frac{3}{4}\times\frac{5}{6};\quad (b)\ \frac{1}{3}\times\frac{3}{6};\quad (c)\ -\frac{1}{5}\times\frac{2}{7};\quad (d)\ \frac{12}{8}\times\frac{10}{60};\quad (e)\ \frac{7}{5}\times5;$

Extra Exercises for Practice

Evaluate

$(a)\ \frac{-1}{-2}\times\frac{3}{5};\ (b)\ \frac{1}{3}\times\frac{6}{2};\ (c)\ -\frac{1}{5}\times\frac{2}{7}\times5;\ (d)\ \frac{4q}{3}\times\frac{6}{-2q};\ (e)\ \frac{4}{5x}\times\frac{3}{-7x}$

1.3 Dividing Fractions

To divide a fraction by another we multiply the first one by the reciprocal (this means turning the fraction upside down) of the second. Put in different words, we invert the second fraction and then multiply the fractions. Let us try a couple of examples.

$$\frac{3}{4}\div\frac{5}{6}=\frac{3}{4}\times\frac{6}{5}=\frac{18}{20}=\frac{9}{10}$$

$$15\div\frac{3}{4}=\frac{15}{1}\times\frac{4}{3}=\frac{60}{3}=20$$

Exercise 1.3

Evaluate the following:

$(a)\ \frac{2}{3}\div\frac{1}{4};\ (b)\ \frac{-5}{6}\div\frac{3}{4};\ (c)\ \frac{2}{-7}\div\frac{4}{21};\ (d)\ -1\div\frac{13}{15};\ (e)\frac{9}{2}\div\frac{9}{0};\ (f)\ \frac{9}{2}\div\frac{0}{9}$

Extra Exercises for Practice

Evaluate

$$(a)\ \frac{4}{5} \div \frac{3}{2};\ (b)\ \frac{2x}{y} \div \frac{x}{2y};\ (c)\ \frac{0}{x} \div \frac{3x}{0};\ (d)\ \frac{a}{b} \div \frac{b}{a};\ (e)\ \frac{p-q}{3} \div \frac{2p-2q}{6}$$

2. Powers and Roots

2.1 Square root and cube root

A square root of a number must be squared (multiply by itself or raised to power 2) to obtain the number. For example:

$\sqrt{4} = 2 \Rightarrow 2$ *is the square root of* $4 \Rightarrow$ *If we square* 2 *i.e* (2^2) *we obtain* 4

$\sqrt{25} = 5 \Rightarrow 5$ *is the square root of* $25 \Rightarrow$ *If we square* 5 *i.e* (5^2) *we obtain* 25

A cube root of a number must be cubed (multiply by itself 3 times or raised to the power 3) to obtain the number. For example:

$\sqrt[3]{27} = 3 \Rightarrow 3$ *is the cube root of* $27 \Rightarrow$ *If we raise* 3 *to the power* 3 *i.e.* (3^3) *we obtain* 27

$\sqrt[3]{64} = 4 \Rightarrow 4$ *is the cube root of* $64 \Rightarrow$ *If we raise* 4 *to the power* 3 *i.e* (4^3) *we obtain* 64

Exercise 2.1

Evaluate the following:

$(a)\ \sqrt{49}$; $(b)\ \sqrt{81}$; $(c)\ \sqrt{144}$; $(d)\ \sqrt{1}$; $(e)\ \sqrt{0}$;

$(f)\ \sqrt[3]{8}$; $(g)\ \sqrt[3]{125}$; $(h)\ \sqrt[3]{1000}$; $(i)\ \sqrt[3]{1}$; $(j)\ \sqrt[3]{0}$

Extra Exercises for Practice

Evaluate

$(a)\ \sqrt{121};\ (b)\ \sqrt{225};\ (c)\ \sqrt{\frac{75}{3}};\ (d)\ \sqrt[3]{216};\ (e)\ \sqrt[3]{-0}$

2.2 Indices Rules

There are generally three rules for indices:

1. Multiplication rule
2. Division rule
3. Power rule

2.2.1 Multiplication Rule

Let us start with some examples:

$$3^2 \times 3^4 = \underbrace{3\times 3}_{3^2}\times \underbrace{3\times 3\times 3\times 3}_{3^4} = 3^6 \Rightarrow 3^2 \times 3^4 = 3^{2+4} = 3^6$$

$$5^3 \times 5^7 = \underbrace{5\times 5\times 5}_{5^3}\times \underbrace{5\times 5\times 5\times 5\times 5\times 5\times 5}_{5^7} = 5^{10} \Rightarrow 5^3 \times 5^7 = 5^{3+7} = 5^{10}$$

From the two examples we have just done, we can see that to multiply two powers of the same number we just add up the powers.

In general: $a^m \times a^n = a^{m+n}$

2.2.2 Division Rule

Again let us try some examples:

$$\frac{3^3}{3^2}=\frac{3\times\cancel{3}\times\cancel{3}}{\cancel{3}\times\cancel{3}}=\frac{3}{1}=3 \; or \; \frac{3^3}{3^2}= 3^3\times3^{-2}=3^{3-2}=3^1=3$$

$$\frac{7^5}{7^3}=\frac{7\times7\times\cancel{7}\times\cancel{7}\times\cancel{7}}{\cancel{7}\times\cancel{7}\times\cancel{7}}=\frac{7\times7}{1}=7^2 \; or \; \frac{7^5}{7^3}=7^5\times7^{-3}=7^{5-3}=7^2$$

$$\frac{1}{3^2}=3^{-2}$$

In general: $\frac{a^p}{a^q}=a^p\times a^{-q}=a^{p-q}$ and $\frac{1}{a^q}=a^{-q}$

2.2.3 Power rule

As before, we should start by doing some examples:

$$(5^2)^3=(5\times5)^3=(5\times5)\times(5\times5)\times(5\times5)=5\times5\times5\times5\times5\times5=5^6 \; or \; (5^2)^3=5^{2\times3}=5^6$$

$$(2^3)^4=(2\times2\times2)^4=(2\times2\times2)\times(2\times2\times2)\times(2\times2\times2)\times(2\times2\times2)$$

$$=\underbrace{2\times2\times2}_{1}\times\underbrace{2\times2\times2}_{2}\times\underbrace{2\times2\times2}_{3}\times\underbrace{2\times2\times2}_{4}=2^{12}$$

$$or \; (2^3)^4=2^{3\times4}=2^{12}$$

In general: $(a^r)^s=a^{r\times s}$

Note: *In addition to these three laws we also need to know that* $a^0=1$, *i.e., apart from zero, any number raised to the power zero is equivalent to one (we can check this using our calculators).*

Exercise 2.2

Evaluate the following:

(a) $3^2\times3$; (b) $5^4\times5^{-1}$; (c) $10^3\times10$

(d) $\frac{2^3}{2^2}$; (e) $\frac{4^4}{4^5}$; (f) $\frac{10^3}{10}$

(g) $(3^2)^4$; (h) $(5^3)^2$; (i) $(6^{-1})^2$

Extra Exercises for Practice

Evaluate

$(a)\ 3^5 \times 3^2;\ (b)\ 2^4 \times 8^3;\ (c)\ 10^3 \times 100^{-3};\ (d)\ \dfrac{2^3}{16^2};\ (e)\ \dfrac{64^2}{4^6}$

ANSWERS

Topic 1

Answer 1.1.1: $(a)\ 7x+2$; $(b)\ -x^2+5xy-3y^2$; $(c)\ x^3-x^2-3x-5$.

Answer 1.1.2 : $a)\ \frac{4}{3}$; $(b)\ \frac{1}{3}$; $(c)\ =\frac{9}{2}$; $(d)\ \frac{16}{9}$ *cannot besimplified*; (e) $\frac{2}{3}$.

Answer 1.1.3: $(a)\frac{3a}{b}$; $(b)\ 3x$; $(c)\ x^2$; $(d)\ x^3$; $(e)\ x^3$.

Answer 1.2.1: **(a)** $5x+15$; **(b)** x^2-6x; **(c)** $15ax-6a$; **(d)** $10y^2-y-21$; **(e)**$-45x^2+36x$; **(f)** $ac+ad+bc+bd$; **(g)** $x^2-7x+10$.

Answer 1.3.1: $(a)\ 5(x-5)$;$(b)\ 9(3x+2)$;$(c)\ -8(x-3)$;$(d)\ x(x+4)$;$(e)\ xy(x+y)$;

$(f)\ 4a(ax-1)$;$(g)\ 8a(3ax+1)$.

Answer 2.1.1(a) 3 ; $(b)\ -12$; $(c)\ 0$; $(d)\ \frac{22}{3}$; $(e)\ 36$; $(f)\ -4$; $(g)\ \frac{9}{2}$; $(h)\ 8$;$(j)\ =0$;

$(k)\ \frac{-9}{10}$; (l) *is not a linear equation because not of the form* $ax+b=0$.

Answer 2.2.1:$(a)\ x=8$ *or* $x-2$;$(b)\ x=-4$ *or* $x=-5$;$(c)\ x=3$ *or* $x=5$;

$(d)\ x=5$ *or* $x=-8$; $(e)\ x=0$ *or* $x=12$.

Answer 2.2.2: $(a)\ -\frac{1}{5}$ *and* $-\frac{4}{5}$; $(b)\ 2$ *and* -1; $(c) -5$*and*-7; $(d)\ \frac{5}{3}$ *and* $\frac{-1}{3}$.

Answer 2.2.3: $(a)\ x=\frac{1}{2}$ *and* $y=4$; $(b)\ x=2$ *and* $y=1$; $(c)\ x=3$*and* $y=-\frac{1}{2}$;

$(d)\ x=-3$ *and* $y=3$.

Answer 3.1: same as in 2.2.3

Topic 2

Answer 1.1:$(a)\ \frac{1}{9}$; $(b)\ \frac{11}{12}$; $(c)\ \frac{34}{15}$; $(d)\ \frac{37}{36}$; $(e)\ \frac{37}{36}$.

Answer 1.2: $(a)\ \frac{5}{8}$; $(b)\ \frac{1}{6}$; $(c)\ -\frac{2}{35}$; $(d)\ \frac{1}{4}$; $(e)\ 7$; $(f)\ =0$.

Answer 1.3: $(a)\ \frac{8}{3}$; $(b)\ \frac{-10}{9}$; $(c)\ -\frac{3}{2}$; $(d)\ \frac{-15}{13}$; $(e)\frac{9}{2}\div\frac{9}{0}$ *impossible*;

(f); *impossible*.

Answer 2.1: $(a)\ \frac{8}{3}$; $(b)\ \frac{-10}{9}$; $(c)\ -\frac{3}{2}$; $(d)\ \frac{-15}{13}$; $(e)\frac{9}{2}\div\frac{9}{0}$ *impossible*;

(f); *impossible*.

Answer 2.2: $(a)\ 3^7$; $(b)\ 5^3$; $(c)\ 10^4$; $(d)\ 2$; $(e)\ 4^{-1}$ *or* $\frac{1}{4}$; $(f)\ 10^2$;

$(g)\ 3^8$; $(h)\ 5^6$; $(i)\ 6^{-2}$ *or* $\frac{1}{6^2}$.

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